Computing Optimal Graphs on Surfaces

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Optimal subgraphs

Given a weighted undirected graph, find an optimal (weighted) subgraph with some desired combinatorial structure.

- Minimum spanning tree
- Shortest paths
- Shortest cycle (girth)
- Perfect matching
- Maximum flow
- Minimum cut
- Hamiltonian path/cycle
- Steiner tree
Optimal geometric graphs

Given a set of points in the plane, find an optimal geometric graph with some desired combinatorial or geometric structure

- closest pair / diameter
- convex hull
- Delaunay triangulation/Voronoi diagram
  - Gabriel graph, EMST, β-skeleta, α-shapes, crust, ...
- minimum weight triangulation
- Euclidean minimum matching, TSP, Steiner tree, ...
- Fermat-Weber problem
Why do we care?

- The problems are easy to state.
- Some are easy; some are hard; some are impossible; some are open.
- The easy problems are incredibly useful for deriving more complex algorithms.
- The hard problems are equally useful for proving lower bounds.
Optimal surface graphs

Given a surface, find an optimal embedded graph with some desired combinatorial, geometric, or topological structure.
What’s a surface?
What’s a surface?
No really, what’s a surface?

A 2-manifold is a topological space in which every point lies in an open neighborhood homeomorphic to the plane.

Equivalently, a 2-manifold is a collection of triangles whose edges are glued together in pairs.

[Dehn, Heegaard ‘07; Radó ‘37]
Combinatorial surface

- An abstract 2-manifold $M$ and a weighted graph $G$ embedded on $M$ so that every face is a disk.

- Consider only curves that are walks in $G$.

- The length of a curve is the sum of the weights of its edges.
Example
Shortest paths

Shortest paths are defined in the standard graph-theoretic sense.

- $O(n \log n)$ time [Dijkstra 53]
- $O(n)$ if genus is $o(n^{1-\varepsilon})$ [Henzinger et al. 97]
Single cycles
Cycles

A cycle is non-contractible if it cannot be continuously deformed to a point.

A simple cycle is non-contractible if it does not bound a disk.

A simple cycle $\gamma$ is non-separating if $M \setminus \gamma$ is connected.
Cycles

non-separating

non-contractible separating

contractible
The 3-path property

For any three paths with the same endpoints, if two of the cycles they form are separating then the third cycle is also separating.
The 3-path property

The shortest non-separating cycle consists of two shortest paths between any pair of antipodal points.

Otherwise, the actual shortest path would create a shorter non-separating cycle.
Shortest nontrivial cycles

3-path property $\Rightarrow O(n^3)$ [Mohar Thomassen 98]
Dijkstra $\Rightarrow O(n^2 \log n)$ [E Har-Peled 02]
$O(n^{3/2} \log n)$ for constant genus [Cabello Mohar 05]

Problem 1: $O(n \log n)$ for constant genus?
Problem 2: $o(n^2)$ in general?
Problem 3: Is it NP-hard to find the shortest non-contractible separating cycle?

- The 3-path property no longer holds.
- The cycle is not necessarily simple.
- Finding a simple cycle that splits the genus exactly in half is NP-hard. [Cabello, Colin de Verdière]
Cut graphs
Cut graphs

A subgraph $H$ of $G$ is a cut graph if $M \setminus H$ is a topological disk.
Shortest cut graph

Computing the shortest cut graph is NP-hard, by a reduction from rectilinear Steiner tree.

[Hanan grid]

[Reference: E Har-Peled 02]
Shortest cut graph

- $O(n^{12g+6b-4})$ time for surfaces with genus $g$ and $b$ boundaries
- $O(\log^2 g)$-approximation in $O(g^2n \log n)$ time.

[Problem 4: Tighten the approximation factor]

[Problem 5: How good an approximation can be computed in $O(n \log n)$ time?]
Partial cut graph

Problem 5: Is it NP-hard to compute the shortest graph $H$ such that $M \setminus H$ has genus 0 (but possibly several boundaries)?
System of loops

A cut graph with one vertex, or equivalently, a set of $2g$ loops through a common basepoint.
Shortest system of loops

The shortest system of loops with a given basepoint can be computed by a simple greedy algorithm in $O(n \log n + k)$ time, where $k=O(gn)$ is the output size.

The shortest overall system of loops can be computed in $O(n^2 \log n)$ time.

[E Whittlesey 05]
Canonical system of loops

A system of loops with a fixed incidence pattern (depending only on the genus)
Canonical system of loops

The shortest system of loops is not necessarily canonical!
Problem 6: How quickly can we construct the shortest canonical system of loops?

A canonical system of loops can be computed in $O(gn)$ time.

[Dey, Schipper 95; Lazarus et al. 01]
Tightening a system of loops

Given a system of loops of complexity $k$, the shortest system in the same homotopy class can be computed in $O(g^4nk^4)$ time.

[Colin de Verdière, Lazarus 02; Colin de Verdière, E 05]

Unfortunately, this does not give us the shortest canonical system!
Homotopy
Homotopy

Two paths are homotopic if one can be continuously deformed into the other, keeping the endpoints fixed at all times.
Shortest homotopic path

Given a path $\pi$ of complexity $k$, the shortest path homotopic to $\pi$ can be computed in time $O(gn(\log n + k))$.  

- Cut the surface into identical polygons with a collection of tight cycles: $O(gn \log n)$
- Glue polygons crossed by the path into a relevant region of the universal cover: $O(gnk)$
- Find shortest path in the relevant region: $O(gnk)$

[Colin de Verdière, E 05]
Hexagonal decomposition
Universal cover
M. C. Escher, Circle Limit IV: Heaven and Hell (1960)
Input path
Relevant lines
The relevant region
Shortest homotopic cycle

- Two cycles are homotopic if one can be continuously deformed into the other.

- The shortest cycle homotopic to a given cycle can be computed in time $O(gnk \log nk)$.  
  [Colin de Verdière, E 05]

- Like shortest homotopic path, but the relevant region is now an annulus.
Pants decompositions
Pants decomposition

Maximal collection of simple, pairwise non-homotopic cycles

Euler’s formula $\Rightarrow 3g-3$ cycles, decomposing the surface into $2g-2$ pairs of pants
Shortest pants decomposition

Problem 8: How quickly can we construct the shortest pants decomposition? Is it NP-hard?

The shortest pants decomposition homotopic to a given pants decomposition can be computed in time $O(g^2nk \log nk)$ by tightening each cycle separately.

[Colin de Verdière, E 05]
Pants clustering

This problem is open even when the “surface” is just the plane minus a finite set of points!

Goal: Minimize the total length of the clustering curves
Problem 8.1: How quickly can we construct the minimum-length pants clustering of a point set? Is it NP-hard?

Assume L₁ or L_∞ metric to avoid nasty numerical problems ⇒ minimum rectilinear pants clustering
Partial results:

- The greedy algorithm fails even in 1D
- If points are collinear, optimal clusters are intervals $\Rightarrow O(n^2)$ time [E, Everett, Lazard, Lazarus]
- $O(1)$-approximation in $O(n \log n)$ time via compressed quadtrees [Eppstein]
- Optimal curves are not necessarily rectangles

□ Problem 8.1.1: Are the optimal curves convex?
“Holy basis”

Maximal non-separating set of non-homotopic cycles, one for each “hole”

(Also half of a homology basis)
Handle decomposition

Maximal set of non-homotopic non-contractible separating cycles, separating the "handles"
Summary

There are lots of interesting graphs that can be defined on any topological surface.

In most cases, we know almost nothing about finding optimal graphs: neither algorithms nor hardness results. Lots of fun open problems.

Are any of these really useful? Who knows? (But then again, who cares?)
Thank you!