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Regular Paper

Unfolding and Dissection of Multiple Cubes, Tetrahedra, and Doubly Covered Squares

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Abstract: In this paper, we propose a new notion of rep-cube which is a net of a cube, and it can be divided into some polygons such that each of them can be folded to a cube. This notion is inspired by *polyomino* and *rep-tile*; both are introduced by Solomon W. Golomb, and well investigated in the recreational mathematics society. We prove that there are infintely many distinct rep-cubes. We also extend this notion to doubly covered squares and regular tetrahedra.

Keywords: Dissection, folding and unfolding, polyomino, rep-cube, rep-tile.

1. Introduction

A *polyomino* is a "simply connected" set of unit squares introduced by Solomon W. Golomb in 1954. Since then, a set of polyominoes has been playing an important role in puzzle society (see, e.g., [7], [9]). In Figure 82 in [7], it is shown that a set of 12 pentominoes exactly covers a cube that is the square root of 10 units on the side. In this context, there are series of results about the set of polyominoes that covers a cube in recreational math society; see [4], [5], [6], [10], [12], [13], [14], [15], [16], [17]. There is a comprehensive survey on a web page maintained by Haubrich [11].

In 1962, Golomb also proposed an interesting notion called "rep-tile": a polygon is a rep-tile of order k if it can be divided into k replicas congruent to one another and similar to the original (see [8], Chap 19).

These notions lead us to the following natural question: is there any polyomino that can be folded to a cube and divided into k polyominoes such that each of them can be folded to a (smaller) cube for some k? That is, a polyomino is a *rep-cube* of order k if it is a net of a cube, and it can be divided into k polyominoes such that each of them can be folded to a cube. If each of these k polyominoes has the same size, we call the original polyomino a *regular* rep-cube of order k. We note that we do not define crease lines for these nets. That is, although each polyomino consists of unit squares, we may fold along the line that is not orthogonal to the unit squares when we fold to a cube. Simple examples can be found in Fig. 1. The first figure indicates two T-shapes that can fold into one cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$. In this net, each T-shape consists of six unit squares and it can fold to a unit cube. On the other hand, gluing these two T-shapes together as shown in the figure, we can fold to a cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$. The dotted lines are not a part of the polyomino which is the net of the cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$, but they are just illustrated to help to understand the squares of size $\sqrt{2} \times \sqrt{2}$.

From this viewpoint, we can find some affirmative examples in the previous results. In [14], [15], [16], we can observe that seven out of eleven developments of a cube have the following property: five copies of each can cover the surface of a cube without overlapping and holes. In our words, there are seven regular rep-cubes of order 5. In [13], Torbijn also investigated the same notion, which was called *cubic hexomino cubes*, and showed some examples for each $k = 4, 5, 7, 9, 10$[*1]. In this paper, we investigate this notion and show more general results. We first give some regular rep-cubes of order k for some specific k. Based on this idea, we give a constructive proof for a series of regular rep-cubes of order $36gk'^2$ for any positive integer k' and an integer g in $\{2, 4, 5, 8, 9, 36, 50, 64\}$. That is, there are infinitely many k that allow regular rep-cube of order k. We also give some non-regular

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*1 In a preliminary version of this paper presented at JCDCGGG, we have not yet found this article which dealt with the same notion.

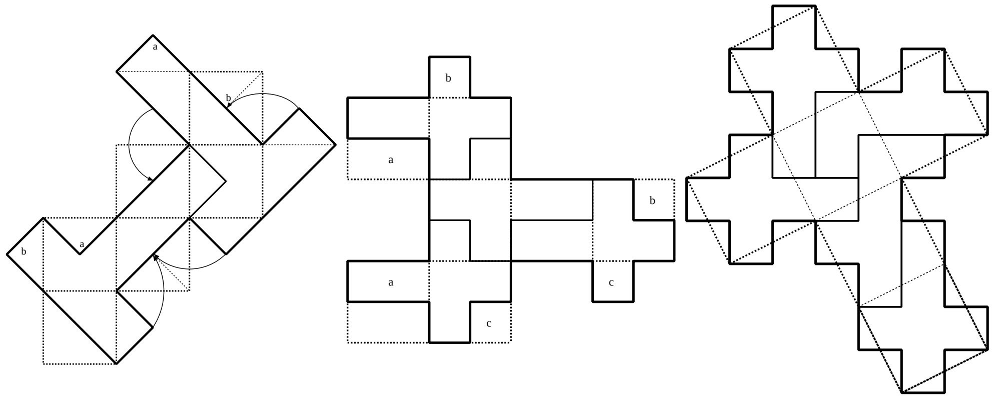

Fig. 1 Rep-cubes of order $k = 2, 4, 5$.

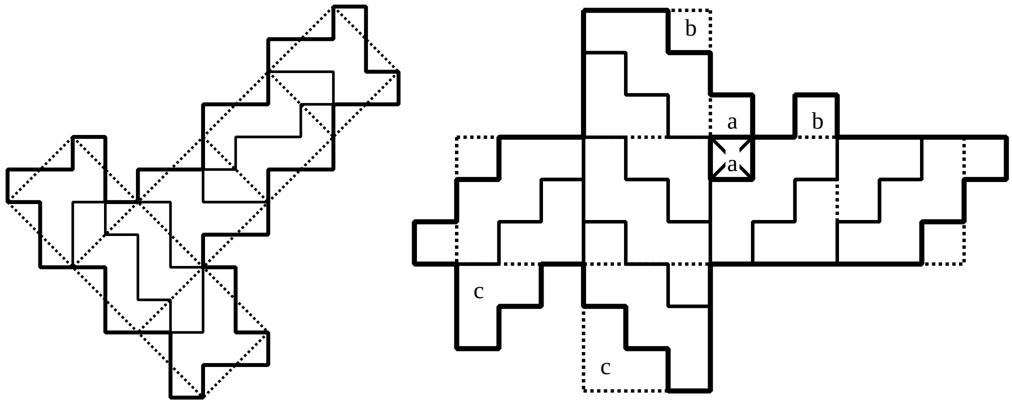

Fig. 2 Rep-cubes of order $k = 8, 9$.

rep-cubes and its variants.

Moreover, we also extend this notion to other dimensions and solids, where each polygon is no longer a polyomino. First, we show the universal result for a doubly covered square. That is, for any positive real numbers $A, a_1, a_2, \ldots, a_k$ such that $\sum_i a_i = A$, there is a net of a doubly-covered square with area A that can be cut into k polygons with areas $a_1, a_2, \ldots, a_k$ such that each of them can be folded into a doubly-covered square. Next, we also show this result can be extended to a regular tetrahedron.

2. Results on Rep-Cubes

We first show some specific solutions.

Theorem 1 There exists a regular rep-cube of order k for $k = 2, 4, 5, 8, 9, 36, 50, 64$.

Proof. For each of $k = 2, 4, 5, 8, 9$, we give a regular rep-cube in Fig.s 1 and 2. It is not difficult to see that they satisfy the condition of rep-cubes.

For $k = 36$, we use six copies of the pattern given in Fig. 3. Using this pattern, we can combine them into any one of eleven nets of a cube.

For $k = 64$, we use one copy of the left pattern in Fig. 4 for the bottom of a big cube, four copies of the center pattern in Fig. 4, and one copy of the right pattern in Fig. 4 for the top of the big cube. The consistency can be easily observed.

For $k = 50$, we make a program for finding packings of nets of unit cubes on twisted grids on bigger cubes by exhaustive search. We found a packing on a $(7, 1)$ twist, i.e., a dissection of the surface of a $\sqrt{50} \times \sqrt{50} \times \sqrt{50}$ cube into 50 nets of unit cubes as shown in Fig. 5. It is worth mentioning that this pattern contains all eleven edge unfoldings of a cube, while the rep-cube of order $k = 9$ in Fig. 2 consists of only one kind of them. □

Based on the solution for $k = 36$ in Theorem 1, we obtain the following theorem:

Theorem 2 There exists a regular rep-cube of order $36gk'^2$ for any positive integer k' and an integer g in $\{2, 4, 5, 8, 9, 36, 50, 64\}$. That is, there exists an infinite number of regular rep-cubes.

Proof. We first choose one pattern in the proof of Theorem 1 according to the value of g. Next we split each unit square in the pattern into k'^2 small squares. Then we replace each small square by the pattern for $k = 36$ in Fig. 3. It is not difficult to see that the notches match with each other since they are arranged properly in the pattern. Therefore, we obtain the theorem. □

One may think that non-regular rep-cubes are more difficult than regular ones. So far, we have found some:

Theorem 3 There exists a non-regular rep-cube of order k for $k = 2, 10$.

Proof. For $k = 2$, the rep-cube is given in Fig. 6(left): this itself folds to a cube of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$, and it can be cut into two pieces such that one folds into a cube of size $2 \times 2 \times 2$, and the other folds into a unit cube. We note that these areas satisfy $6 \times (\sqrt{5})^2 = 6 \times 1^2 + 6 \times 2^2 = 30$.

For $k = 10$, the rep-cube is given in Fig. 6(right): this pattern contains 150 unit squares. It is easy to see that nice nets of unit cube use 54 unit squares in total. The remaining 96 squares form a net of cube of size $4 \times 4 \times 4$. Moreover, this

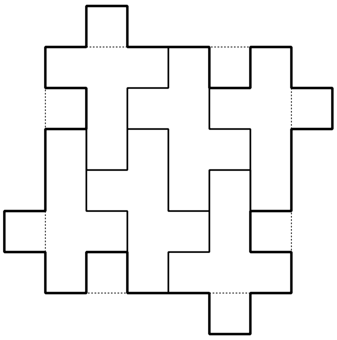

Fig. 3 Pattern for rep-cubes of order $k = 36$.

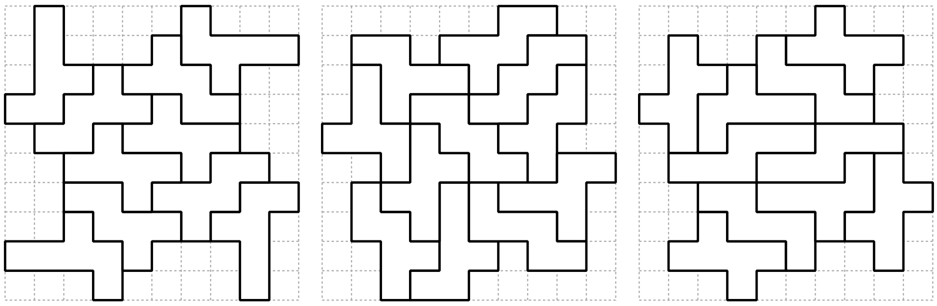

Fig. 4 Patterns for rep-cubes of order $k = 64$.

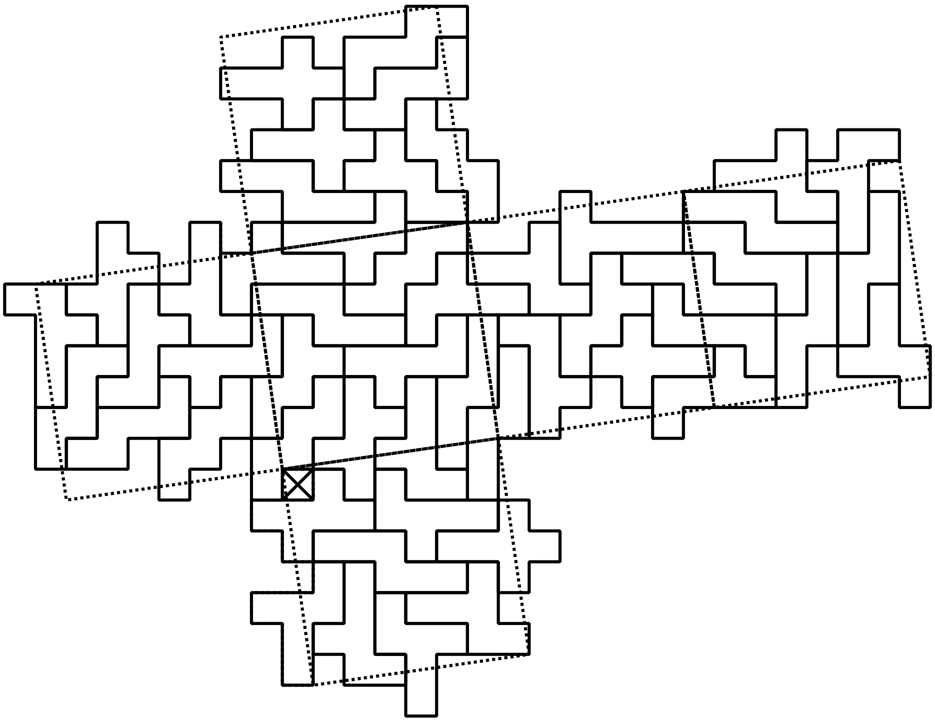

Fig. 5 Pattern for rep-cubes of order $k = 50$.

pattern also folds to a cube of size $5 \times 5 \times 5$. These areas satisfy $150 = 6 \times 5^2 = 6 \times (3^2) + 6 \times (4^2) = 6(3^2 + 4^2)$. □

3. Generalization

One natural extension of the notion of the rep-cube is a different dimension. We first focus on the 2 dimensional case; doubly-covered squares. A *doubly-covered square* consists of two unit squares such that every two corresponding edges of the two squares are glued to each other. A unit doubly covered square has volume 0 and area 2.

Before considering doubly-covered squares, we first show a useful lemma:

Lemma 4 Let P be a cylinder of circumference a and height b. Then, for any $0 < \theta \leq 90°$, we have a common development of P and the other cylinder Q that has circumference $x/2$ and height y with $x = \frac{b}{\sin\theta}$ and $y = a\sin\theta$.

Proof. We give a construction of Q in Fig. 7. First, we cut the dotted line in Fig. 7(1). Then we have a parallelogram of edge lengths a and $x = \frac{b}{\sin\theta}$. Rolling it up along the edge of length x, we have a cylinder of desired size in Fig. 7(3). □

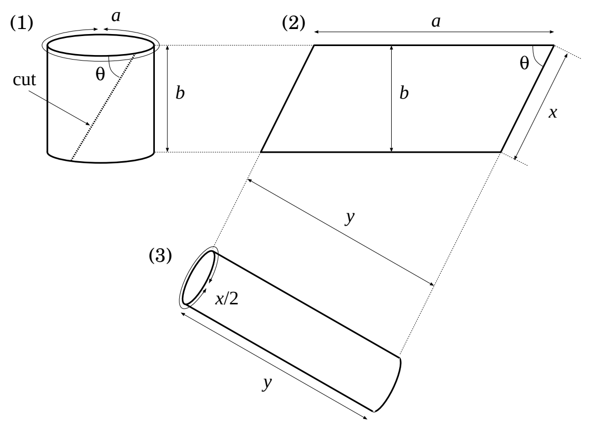

Fig. 7 (1) A cylinder of circumference a and height b, (2) a common development of two cylinders, (3) the other cylinder of circumference $x/2$ and height y.

We note that by changing θ from 90° to any small angle greater than 0°, we can have any long x greater than or equal to $b/2$.

For doubly-covered squares, we have the following theorem:

Theorem 5 For any positive real numbers $A, a_1, a_2, \ldots, a_k$

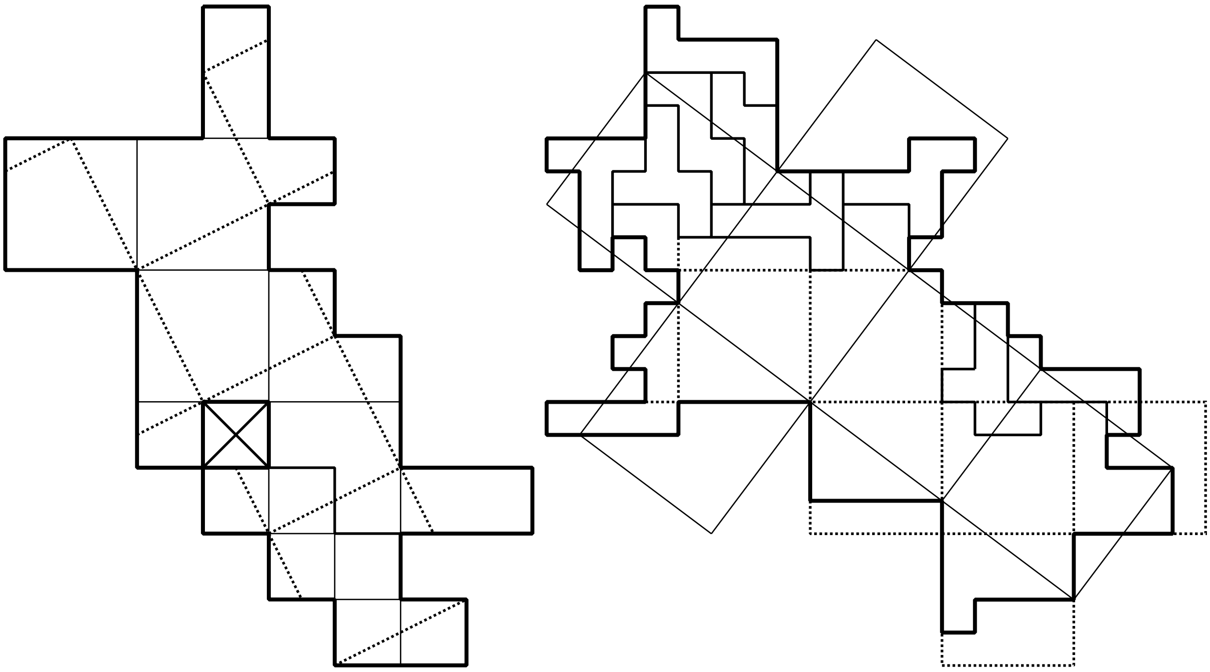

Fig. 6 Patterns for non-regular rep-cubes of order $k = 2, 10$.

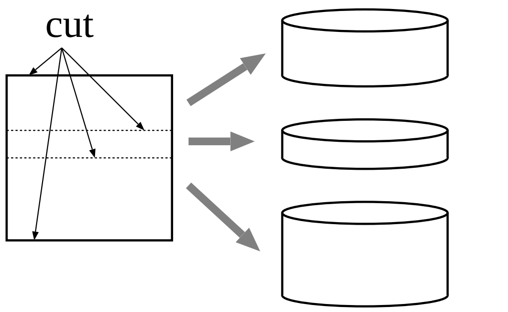

Fig. 8 One doubly-covered square to three doubly-covered squares.

such that $\sum_i a_i = A$, there is a net of a doubly-covered square with area A that can be cut into k polygons with areas $a_1, a_2, \ldots, a_k$, each of which can be folded into a doubly-covered square.

Proof. We first split the doubly-covered square of area A into k pieces along horizontal lines so that each piece has area $a_1, a_2, \ldots, a_k$ (see Fig. 8). After the split, we have two pieces of envelope shapes of area a_1 and a_k, and $k-2$ pieces of cylindrical paper strip of area $a_2, \ldots, a_{k-1}$. We cut two more lines to make two envelope shapes into cylindrical paper strips as well.

Now, we consider the ith strip of area a_k. Its circumference is $2(\sqrt{A/2}) = \sqrt{2A}$, and hence its height is $a_k/\sqrt{2A}$. It is easy to see that $a_k/\sqrt{2A} < \sqrt{a_k/2}$ since $a_k < A$. Therefore, we can apply Lemma 4 to obtain a common development of this ith cylinder and another cylinder of circumference $2\sqrt{a_k/2}$ and of height $\sqrt{a_k/2}$, which can be glued to a doubly covered square easily. □

The trick in Theorem 5 also works for regular tetrahedra:

Theorem 6 For any positive real numbers $A, a_1, \ldots, a_k$ such that $\sum_i a_i = A$, there is a net of a regular tetrahedron with area A that can be cut into k polygons with areas $a_1, \ldots, a_k$, each of which can be folded into a regular tetrahedron.

Proof. It is known that a tetrahedron of area 4 can be folded by any parallelogram of base of length 2 and height $\sqrt{3}/2$ (see Fig. 9; the detailed characterization of a regular tetrahedron is given by Akiyama and Nara [3]).

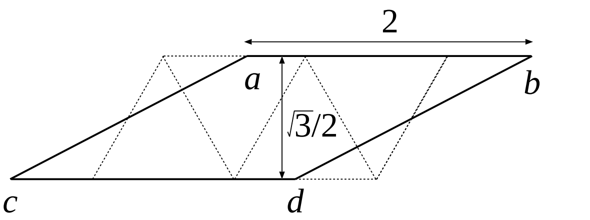

Fig. 9 Any parallelogram of base of length 2 and height $\sqrt{3}/2$ can fold to a regular tetrahedron; first, glue the edge ac and bd, and squash the resulting cylinder along the dotted lines.

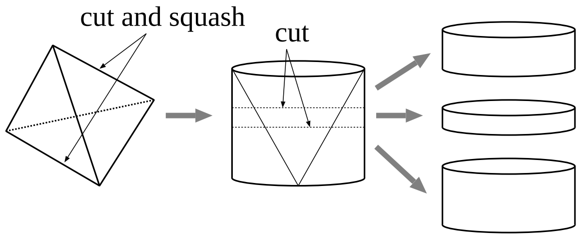

Fig. 10 One regular tetrahedron to three cylinders; each of them can fold to a regular tetrahedron.

When we cut two skew edges of a regular tetrahedron of area A, we obtain a cylinder of circumference $\sqrt{\frac{4A}{\sqrt{3}}}$ and of height $\sqrt{\frac{\sqrt{3}A}{4}}$ (Fig. 10).

Thus, using the same method in the proof of Theorem 5, we obtain the theorem. □

4. Conclusion

In this paper, we introduce a new notion of "rep-cube," and show several examples. So far, we have no systematic ways to investigate them. However, from the trivial constraint for the areas, we can consider many variants as shown in the last example for

$k = 10$: Is there a rep-cube of order 6 from a $3 \times 3 \times 3$ cube into one $2 \times 2 \times 2$ cube and five $1 \times 1 \times 1$ cubes, and so on? Especially, one interesting open question is whether there is a rep-cube of order 2 from one $5 \times 5 \times 5$ cube into one $4 \times 4 \times 4$ cube and $3 \times 3 \times 3$ cube. We note that this size comes from the Pythagoras triangle $3^2 + 4^2 = 5^2$. We have already known that there are infinitely many Pythagoras triangles. For each of them, can we construct a rep-cube of order 2?

Is there any integer k such that we have no regular rep-cube of order k? It seems to be unlikely that there is a regular rep-cube of order 3. How can we prove that? In this paper, we also introduce "regular" rep-cubes. One natural additional condition may be making every small development congruent; for example, each example for $k = 2, 4, 9$ satisfies this condition. What happens if we employ this additional condition?

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