Chapter 4

Spherical Degeneracies

The *spherical degeneracy problem* asks, given n points in $\mathbb{R}^d$, if any $d+2$ lie on the same sphere. This problem can be transformed into the affine degeneracy problem one dimension higher by projecting the input vertically onto the paraboloid $x_{d+1} = x_1^2 + \cdots + x_d^2$. The images of cospherical points in $\mathbb{R}^d$ under this projection lie on a single hyperplane in $\mathbb{R}^{d+1}$. Furthermore, if the point q lies inside (resp. outside) the sphere defined by $d+1$ points $p_0, \ldots, p_d$ in $\mathbb{R}^d$, then the image of q lies below (resp. above) the hyperplane in $\mathbb{R}^{d+1}$ defined by the images of $p_0, \ldots, p_d$ [62]. Sidedness queries on the lifted point set are thus equivalent to *insphere queries* in the original d-dimensional point set. Two-dimensional insphere queries are also called *incircle queries*. Algebraically, the result of an insphere query is given by the sign of the following determinant.

$$\begin{vmatrix} 1 & p_{01} & p_{02} & \cdots & p_{0d} & \sum_i p_{0i}^2 \\ 1 & p_{11} & p_{12} & \cdots & p_{1d} & \sum_i p_{1i}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & p_{d1} & p_{d2} & \cdots & p_{dd} & \sum_i p_{di}^2 \\ 1 & q_1 & q_2 & \cdots & q_d & \sum_i q_i^2 \end{vmatrix}$$

A special case of the spherical degeneracy problem ignores $(d+2)$-tuples that lie on spheres of infinite radius (*i.e.*, hyperplanes). We refer to any $(d+2)$-tuple that lies on a sphere of finite radius as a *proper* spherical degeneracy.

In this chapter, we show that $\Omega(n^3)$ incircle queries are required to detect circular degeneracies in the plane, and $\Omega(n^{d+1})$ insphere queries are required to detect proper spherical degeneracies in $\mathbb{R}^d$. Both lower bounds are tight [65, 68, 69]. Like our previous results, the lower bounds in this chapter are based on adversary arguments using "collapsible tuples".

4.1 Circular Degeneracies

Theorem 4.1. *Any decision tree algorithm that detects proper circular degeneracies, using only incircle queries, must have depth $\Omega(n^3)$.*

Proof: The adversary presents the following set of points:

$$S \triangleq \bigcup_{i=1}^{n/6} \left\{ (2^{5i+2}, 0), (2^{5i-2}, 0), (0, 2^{5i}) \right\} \cup \bigcup_{i=1}^{n/2} \left\{ (0, 2^{5(i-n/6)+1}) \right\}$$

The set consists of four subsets of points, two contained in each positive coordinate axis. We easily verify that this set contains no proper circular degeneracies, using the fact that four points $(a, 0)$, $(b, 0)$, $(0, c)$, and $(0, d)$ are cocircular if and only if $ab = cd$.

For each $1 \leq i, j, k \leq n/6$, the following points are "almost" cocircular.

$$(2^{5i+2}, 0), (2^{5j-2}, 0), (0, 2^{5k}), (0, 2^{5(i+j-k)+1})$$

Each such set of points is a *collapsible 4-tuple*. The adversary can collapse any such tuple by changing the four points to the following.

$$(2^{5i+3/2}, 0), (2^{5j-3/2}, 0), (0, 2^{5k-1/2}), (0, 2^{5(i+j-k)+1/2})$$

We easily verify that this change does not introduce any other new circular degeneracies or change the result of any other incircle query. There are $n^3/216 = \Omega(n^3)$ collapsible 4-tuples, each of which must be checked by the algorithm. □

Since collapsing a 4-tuple preserves both the coordinate orders of the points and their order type, we immediately have the following stronger theorem.

Theorem 4.2. *Any decision tree algorithm that detects proper circular degeneracies, using only incircle queries, sidedness queries, and coordinate comparisons, must perform $\Omega(n^3)$ incircle queries in the worst case.*

A lower bound for the general problem follows from a very simple argument, similar to the weird moment curve argument used throughout the last two chapters. In this case, the "weird" curve we need is a parabola.

Theorem 4.3. *Any decision tree that decides whether n points in $\mathbb{R}^2$ is circularly degenerate, using only incircle queries, must have depth $\Omega(n^3)$.*

Proof: Four points $(a, a^2), (b, b^2), (c, c^2), (d, d^2)$ on the unit parabola are cocircular if and only if $a+b+c+d = 0$. (Indeed, the paraboloid lifting function $(x, y) \mapsto (x, y, x^2+y^2)$ maps the unit parabola to a skewed three-dimensional weird moment curve; see Figure 3.1!) Let X be the set of integers from $-n$ to n. There are clearly $\Theta(n^3)$ 4-tuples in X whose sums are zero. The adversary presents a set of points on the unit parabola with x-coordinates taken from the set $X + 1/8$. This set is nondegenerate and has $\Omega(n^3)$ collapsible 4-tuples. $\square$

We can extend the model of computation in a similar fashion as in Section 2.3, but with a different set of primitives. A *linear fractional transformation* of the plane (or more formally, of the Riemann sphere $\mathbb{CP}^1$) is any transformation that maps circles to circles. If we represent the points of $\mathbb{R}^2$ in complex homogeneous coordinates — representing $(x, y) \in \mathbb{R}^2$ by any complex multiple of $(1 + 0i, x + yi) \in \mathbb{C}^2$ — then a linear fractional transformation is equivalent to a linear transformation of $\mathbb{C}^2$.

We say that a query is *circularly allowable* if some linear fractional transformation of the set (X, X^2) is nondegenerate with respect to that query, where $X = \{-n, 1-n, \ldots, n\}$ is the set of numbers described in the proof of Theorem 4.3. Circularly allowable queries include first- and second-order coordinate comparisons and sidedness queries, but do not include comparisons between arbitrary incircle determinants.

Arguments similar to those in Section 2.3 give us the following theorem.

Theorem 4.4. *Any decision tree that decides whether n points in $\mathbb{R}^2$ is circularly degenerate, using only incircle queries and a finite number of circularly allowable queries, requires $\Omega(n^3)$ incircle queries in the worst case.*

4.2 Proper Spherical Degeneracies

In order to extend Theorem 4.1 to the d-dimensional case, we exhibit a set S of $O(n)$ points in $\mathbb{R}^d$ that contains $\Omega(n^{d+1})$ *collapsible* $(d+2)$*-tuples*: sets of $d+2$ non-cospherical points in S that can be moved so that they become cospherical, without changing the result of any other insphere query. The following construction is primarily due to Raimund Seidel [73].

The point set S in question is the union of $d+1$ smaller sets, $S_1 \cup \cdots \cup S_d \cup D$, where each S_i consists of $n/2$ even integer points on the positive x_i-axis, and D consists of about $(d+1)n/2$ "odd" points on the main diagonal $(t, \ldots, t)$. At the risk of confusing the reader, we let each subscripted variable t_i refer simultaneously to a point on the x_i-axis and that point's non-zero coordinate. Similarly, each unsubscripted variable t refers simultaneously to a point on the main diagonal and the value of all its coordinates.

To make our construction precise, the sets S_i include points t_i such that t_i is even and

$$a_i < t_i \leq a_i + n,$$

where $a_1 = 0$, a_i is large for all $1 < i < d$ (say $a_i = n^3 + in$), and a_d is huge (say $a_d = 2^n$). The set D includes points t such that dt is odd and

$$A < dt \leq A + (d+1)n,$$

where $A = \sum_{i=1}^{d} a_i$.

Lemma 4.5. *The set S contains no proper spherical degeneracies.*

Proof: For all $1 \leq i \leq d$, let t_i and t_i' be two distinct points in S_i, and let t and t' be two distinct points in D. Note that with our choice of values for a_i we have the following bounds.

$$\begin{array}{rcccl} -(d+1)n & < & t_1 + \cdots + t_d - dt & < & dn \\ 1/n & < & \dfrac{1}{t_1} + \cdots + \dfrac{1}{t_d} - \dfrac{1}{t} & < & 1 \\ \dfrac{d-2}{n^3} - o\left(\dfrac{1}{n^3}\right) & < & \dfrac{1}{t_2} + \cdots + \dfrac{1}{t_d} - \dfrac{1}{t} & < & \dfrac{d-2}{n^3} \end{array}$$

By examining the appropriate insphere determinants, we find that the cosphericity of any set of d+2 points from S is expressed by the vanishing of one of the following algebraic expressions.

- Two points from the x_i-axis, one from each of the other axes, and one from the main diagonal:

$$t_1 + \cdots + t_d - dt + t_i t_i' \left(\frac{1}{t_1} + \cdots + \frac{1}{t_d} - \frac{1}{t} \right) \tag{4.1}$$

- Two points from the main diagonal, and one from each axis:

$$t_1 + \cdots + t_d - dt + dtt' \left(\frac{1}{t_1} + \cdots + \frac{1}{t_d} - \frac{1}{t} \right) \tag{4.2}$$

- Two points from the x_i axis, two points from the x_j axis, and $d-2$ points elsewhere:

$$t_i t_i' - t_j t_j' \tag{4.3}$$

- Two points from the x_i axis, two points from the main diagonal, and $d-2$ points elsewhere:

$$t_i t_i' - dtt' \tag{4.4}$$

With t_i, t_i', t, t' chosen in the indicated ranges and with the indicated parities, expression (4.1) never vanishes, since the last term dominates when $i > 1$, and the whole expression differs from an odd integer by less than d/n when $i = 1$. Expression (4.2) never vanishes, since the last term always dominates. Expression (4.3) never vanishes, since the x_i-range and the x_j-range are disjoint. Finally, expression (4.4) never vanishes, since the second term dominates when $i < d$, and the first term dominates when $i = d$. □

Lemma 4.6. *The set S contains $\Omega(n^{d+1})$ collapsible $(d+2)$-tuples.*

Proof: For any choice of two distinct points t_1, t_1' from S_1 and one point t_i from each of the other S_i, we can choose the point from D with all coordinates equal to $(t_1 + \cdots + t_d + t_1' - 1)/d$, so that these points form a collapsible $(d+2)$-tuple. To collapse the tuple, the adversary decreases the non-zero coordinates of the axis points by $1/(2d+2)$ and increases each coordinate of the main diagonal point by just under $1/2d + 1/n$. □

Our previous adversary argument immediately implies the following lower bound.

Theorem 4.7. *Any decision tree algorithm that detects proper spherical degeneracies in $\mathbb{R}^d$, using only insphere queries, must have depth $\Omega(n^{d+1})$.*

As we did in the planar case, we can extend this lower bound to allow additional computational primitives.

Theorem 4.8. *Any decision tree algorithm that detects proper spherical degeneracies in $\mathbb{R}^d$, using only insphere queries, sidedness queries, and coordinate comparisons, must perform $\Omega(n^{d+1})$ insphere queries in the worst case.*

Proof: It suffices to show that collapsing a $(d+2)$-tuple does not change the result of any coordinate comparison or sidedness query. Coordinate comparisons don't change, since the order of the points is preserved within each subset, and the range of coordinates for the points on the main diagonal is disjoint from the range of coordinates for the points on any coordinate axis.

By examining the appropriate sidedness determinants, we find simple algebraic expressions giving the orientation of any simplex in S, similar to the expressions (4.1)–(4.4) describing cosphericity. There are only three nontrivial cases.

- Two points on one axis, and no points on one axis or the main diagonal:

$$t_i - t_i' \tag{4.5}$$

- Two points on the main diagonal, and no points on one axis:

$$t - t' \tag{4.6}$$

- One point on each axis, and one on the main diagonal:

$$\frac{1}{t_1} + \cdots + \frac{1}{t_d} - \frac{1}{t} \tag{4.7}$$

In every other case, the simplex is always degenerate.

In the first and second cases, sidedness queries reduce to coordinate comparisons. In the original configuration S, expression (4.7) is positive, and collapsing a tuple only makes it bigger, since each t_i is decreasing and t is increasing. Thus, no simplex in S changes orientation. □

4.3 Open Problems

We conjecture that $\Omega(n^{d+1})$ insphere queries are required to detect *arbitrary* spherical degeneracies. (I claimed this lower bound in [73], but my "proof" was incorrect.) A proof of this conjecture would follow immediately from the construction of a set of numbers having $\Omega(n^{d+1})$ $(d+2)$-tuples in the zeroset of a certain symmetric polynomial, by applying the same "sliding adversary" argument used to prove many of our previous lower bounds. For example, in three dimensions, we need $\Omega(n^4)$ 5-tuples in the zeroset of the polynomial

$$1 + \sum_{1 \le i \le j \le 5} t_i t_j.$$

Unlike all our previous constructions, the adversary set we used to prove Theorem 4.7 is not obtained by perturbing a highly degenerate point configuration. Is there a set of n points in $\mathbb{R}^d$ with $\Omega(n^{d+1})$ independent spherical degeneracies, for any $d \geq 3$? Such a set might lead to a lower bound for the general spherical degeneracy problem, and it might also allow us to define a general class of "spherically allowable" queries, strengthening Theorem 4.8.

"... In that blessed region of Four Dimensions, shall we linger on the threshold of the Fifth, and not enter therein? Ah, no! Let us rather resolve that our ambition shall soar with our corporal ascent. Then, yielding to our intellectual onset, the gates of the Sixth Dimension shall fly open; after that a Seventh, and then an Eighth —"

How long I should have continued I know not. In vain did the Sphere, in his voice of thunder, reiterate his command of silence, and threaten me with the direst penalties if I persisted.

— Edwin Abbott, *Flatland*, 1884