Greedy Optimal Homotopy and Homology Generators

Jeff Erickson and Kim Whittlesey
University of Illinois, Urbana-Champaign

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What's the problem?

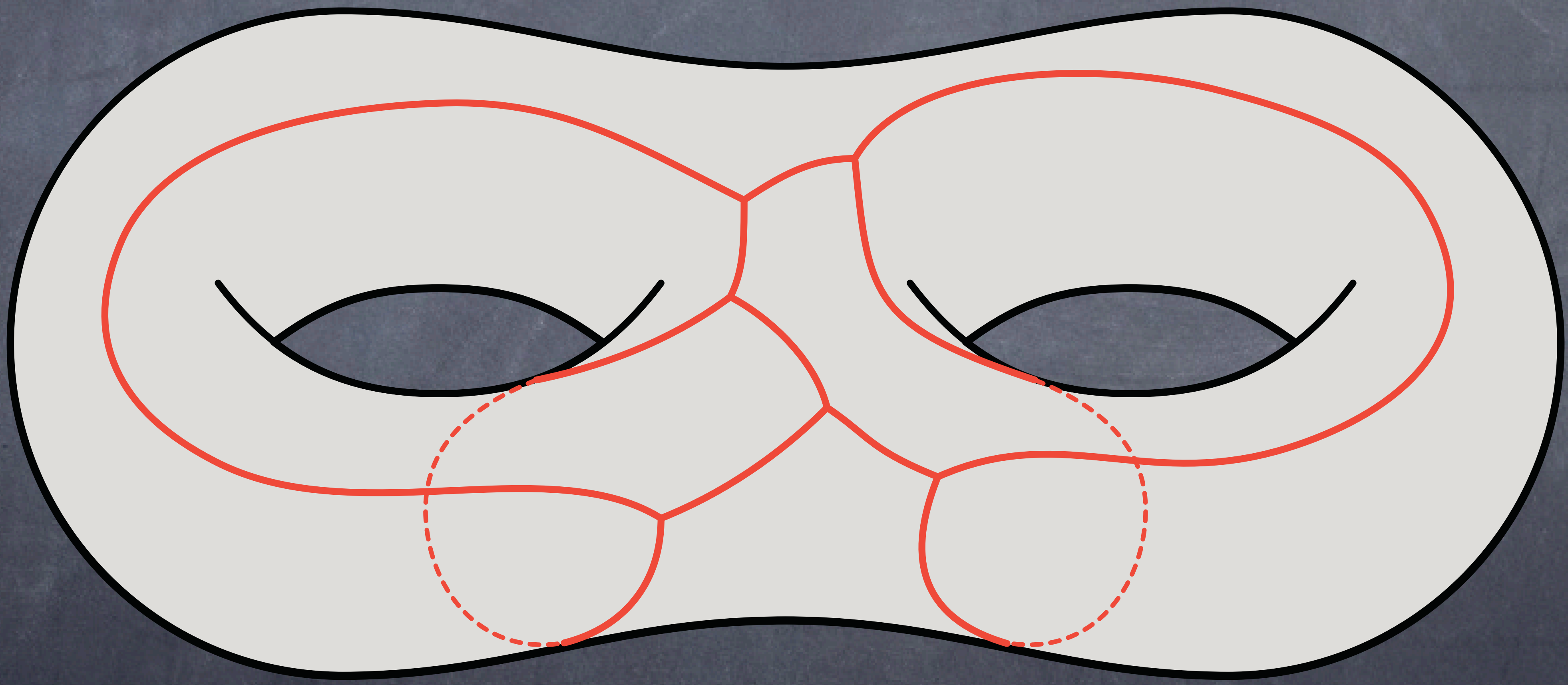
Given a topologically complex surface, cut it into one or more topological disks.

- Simplification, remeshing, compression, texture mapping, parameterization, . . .
- Computing separators and tree decompositions of non-planar graphs

Moreover, cut the surface as little as possible.

Cut Graph

An embedded graph whose removal cuts the surface into a single topological disk.



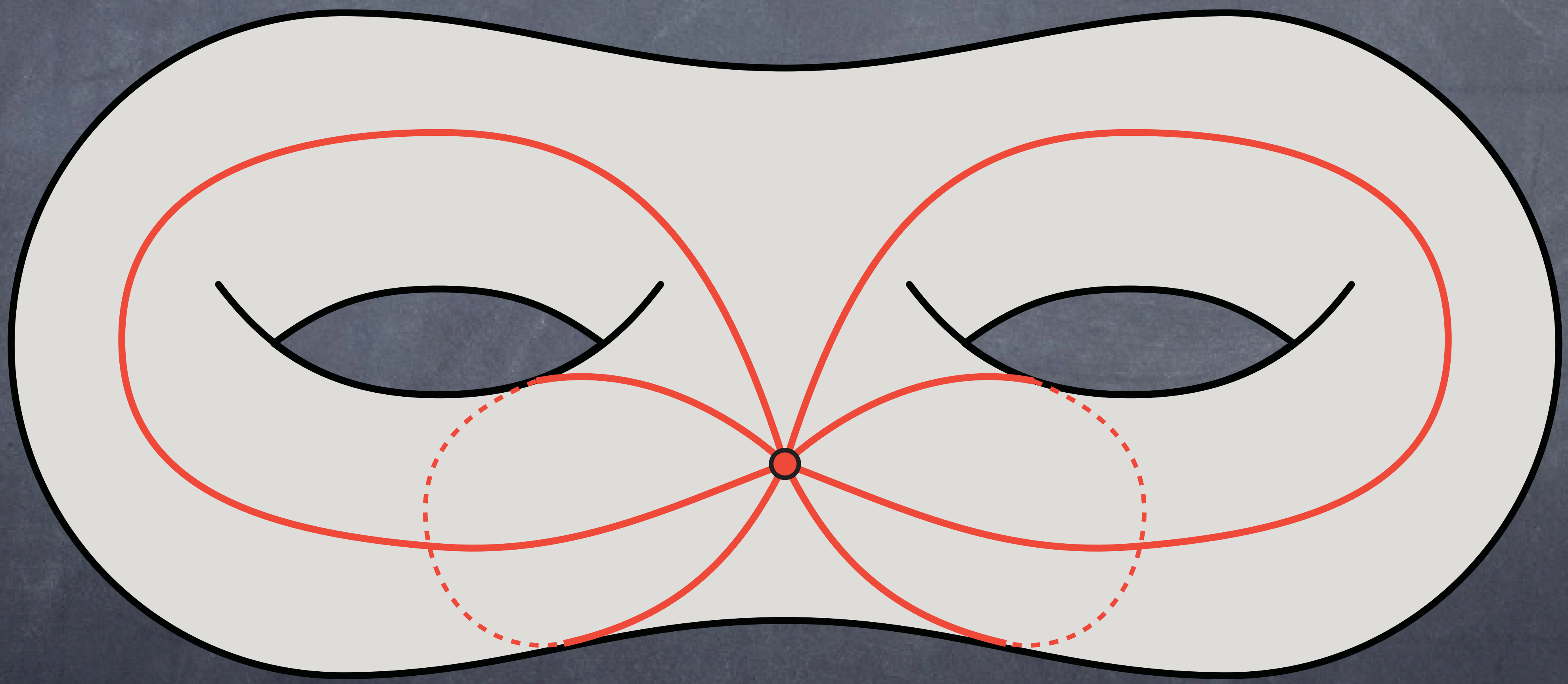
Optimality is hard

Erickson, Har-Peled 2000:

- Computing the minimum-length cut graph for a given surface is NP-hard.
- A $O(\log^2 g)$-approximation can be found in $O(g^2 n \log n)$ time, where g is the genus of the surface.

System of loops

A cut graph with one vertex, or equivalently, a set of 2g loops through a common basepoint



Optimality is not so hard?

Colin de Verdière, Lazarus 2002:

- Given a system of loops on a surface, the shortest system of loops in the same homotopy class can be computed in (large) polynomial time.*
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*under some mild (and probably unnecessary) assumptions.

Optimality is easy!

Our main result:

- The shortest system of loops with a given basepoint can be computed in $O(n \log n)$ time using a simple greedy algorithm.
- The overall shortest system of loops can be computed in $O(n^2 \log n)$ time.
- More machinery improves both running times by a factor of $O(\log n)$ if $g=O(n^{0.999})$.

Outline

- Definitions: combinatorial surface, homotopy, homotopy basis, exponent sums, greedy homotopy basis, homology

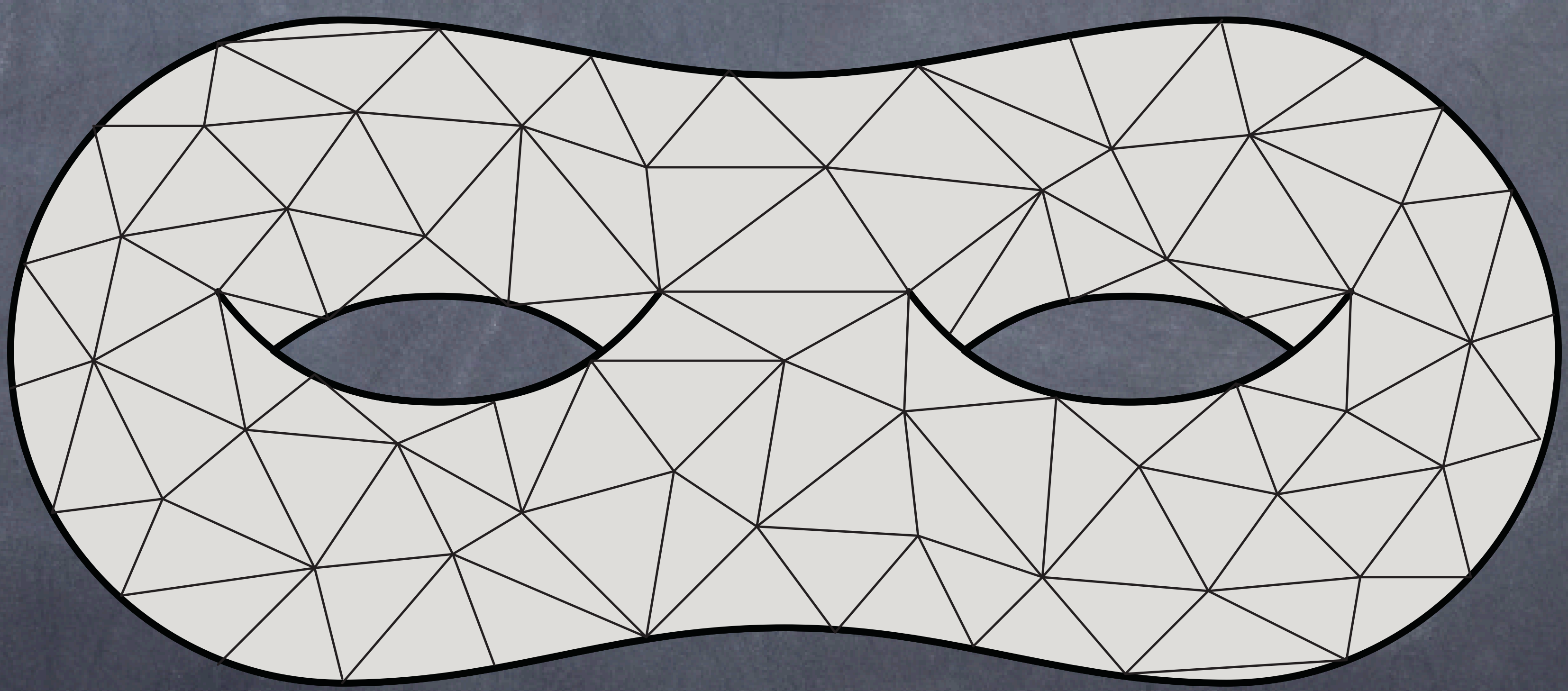
- Proof of optimality

- Implementation: dual graph, shortest paths + max spanning tree

Surfaces and topology

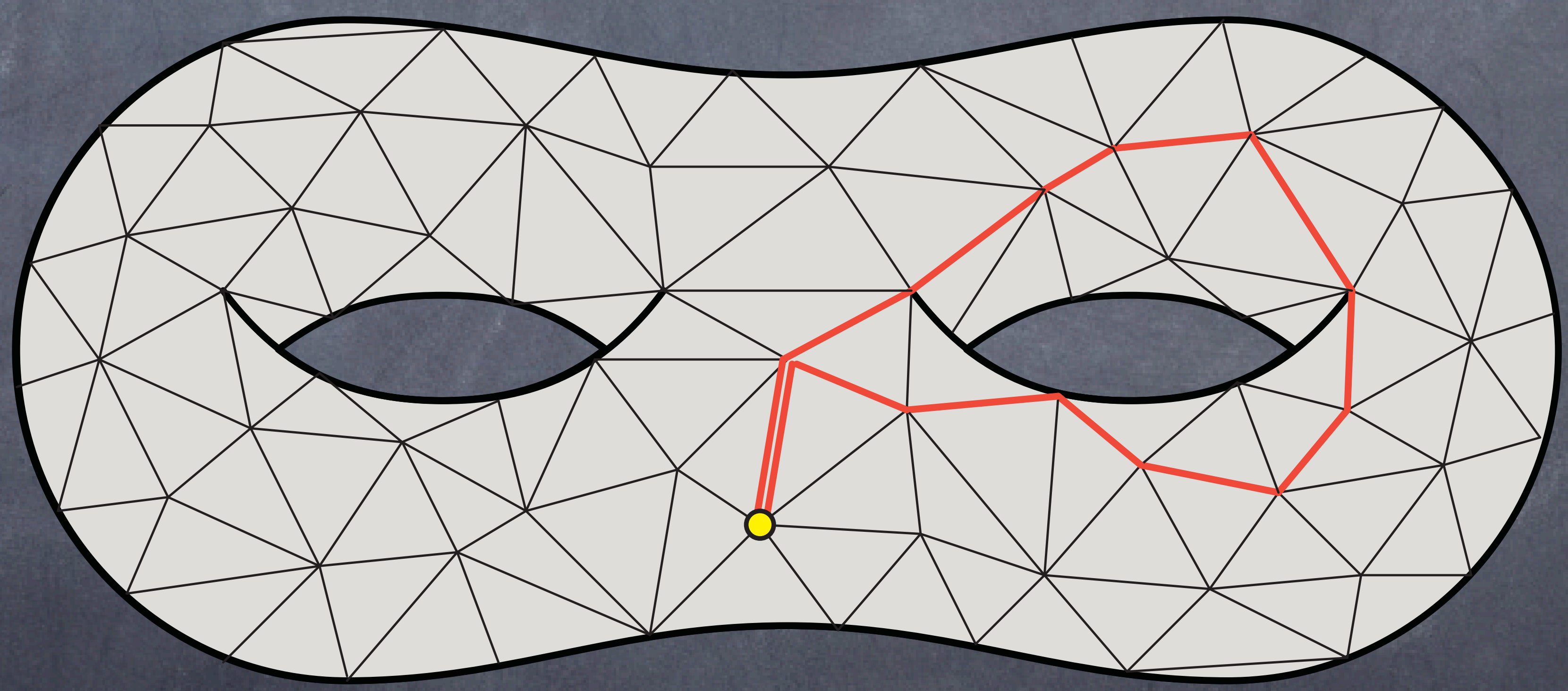
Combinatorial surface

An abstract compact orientable 2-manifold with a weighted graph G, embedded so that every face is a disk.



Loop

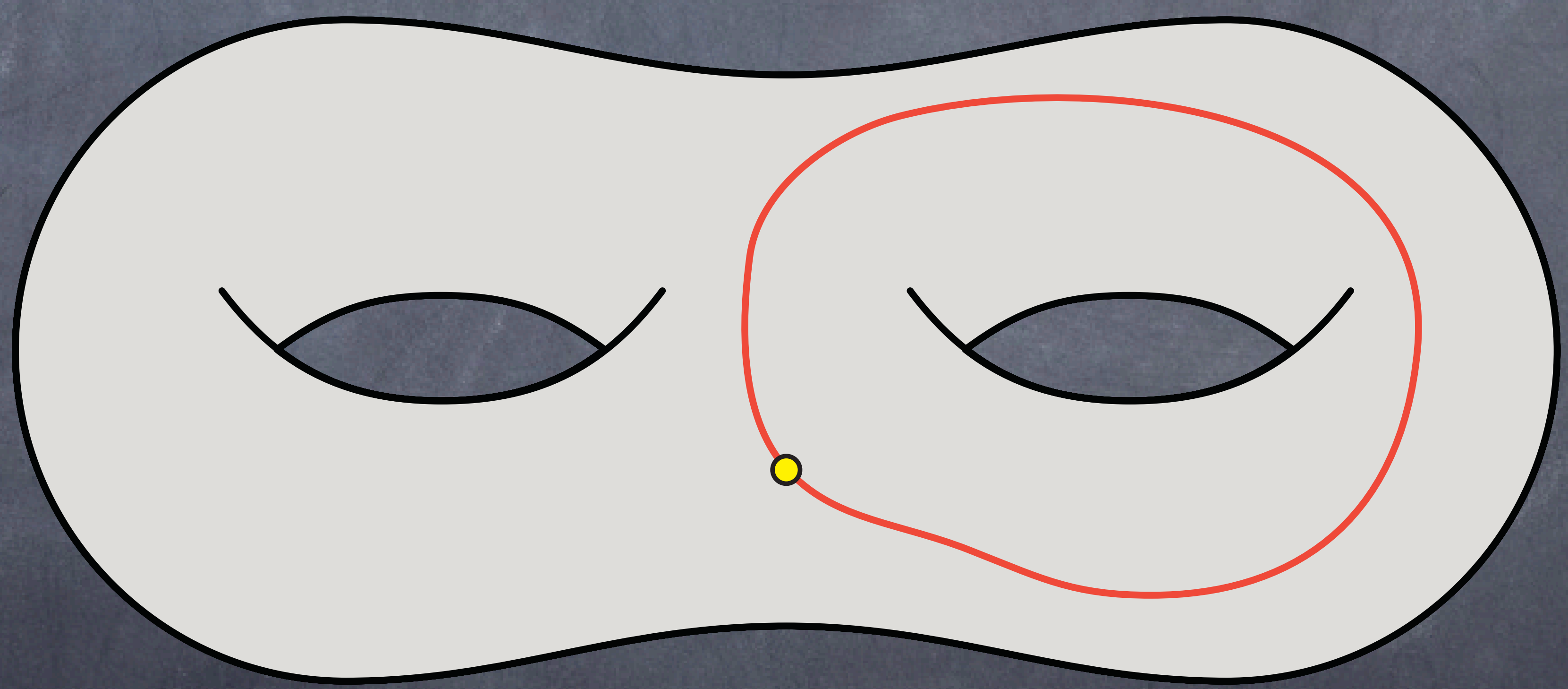
The image of a function $L:[0,1] \to G$ such that $L(0) = L(1) = x$, where x is a fixed basepoint.



Our algorithms consider only circuits in G.

Homotopy

Two loops L and L′ are homotopic (written $L \simeq L'$) if one loop can be continuously deformed into the other.



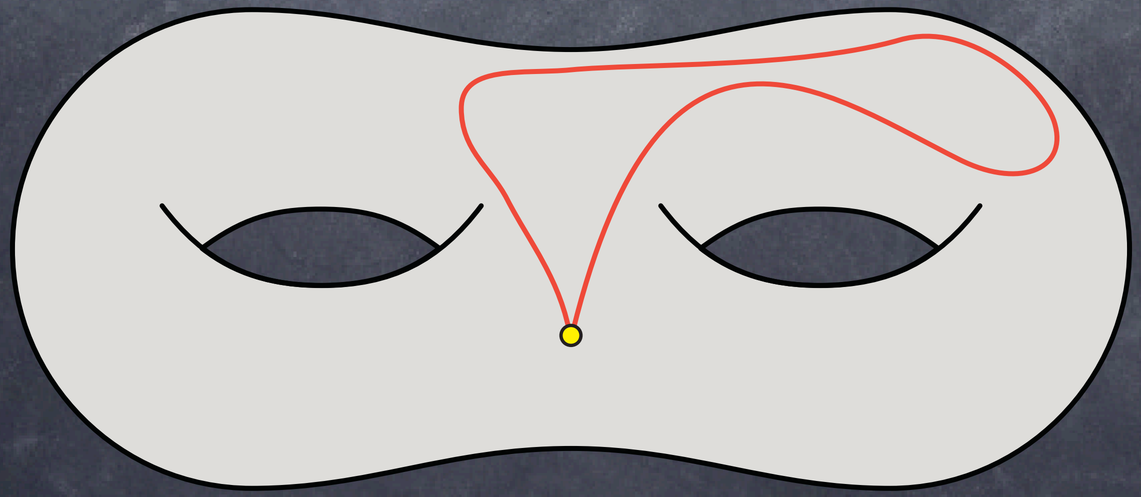
Formally, a homotopy from L to L′ is a continuous function $h:[0,1]\times[0,1] \to M$ where $h(0,t)=L(t)$, $h(1,t)=L'(t)$, and $h(s,0)=h(s,1)=x$

The fundamental group

The set of homotopy classes form a group $\pi_1(M,x)$ under concatenation:

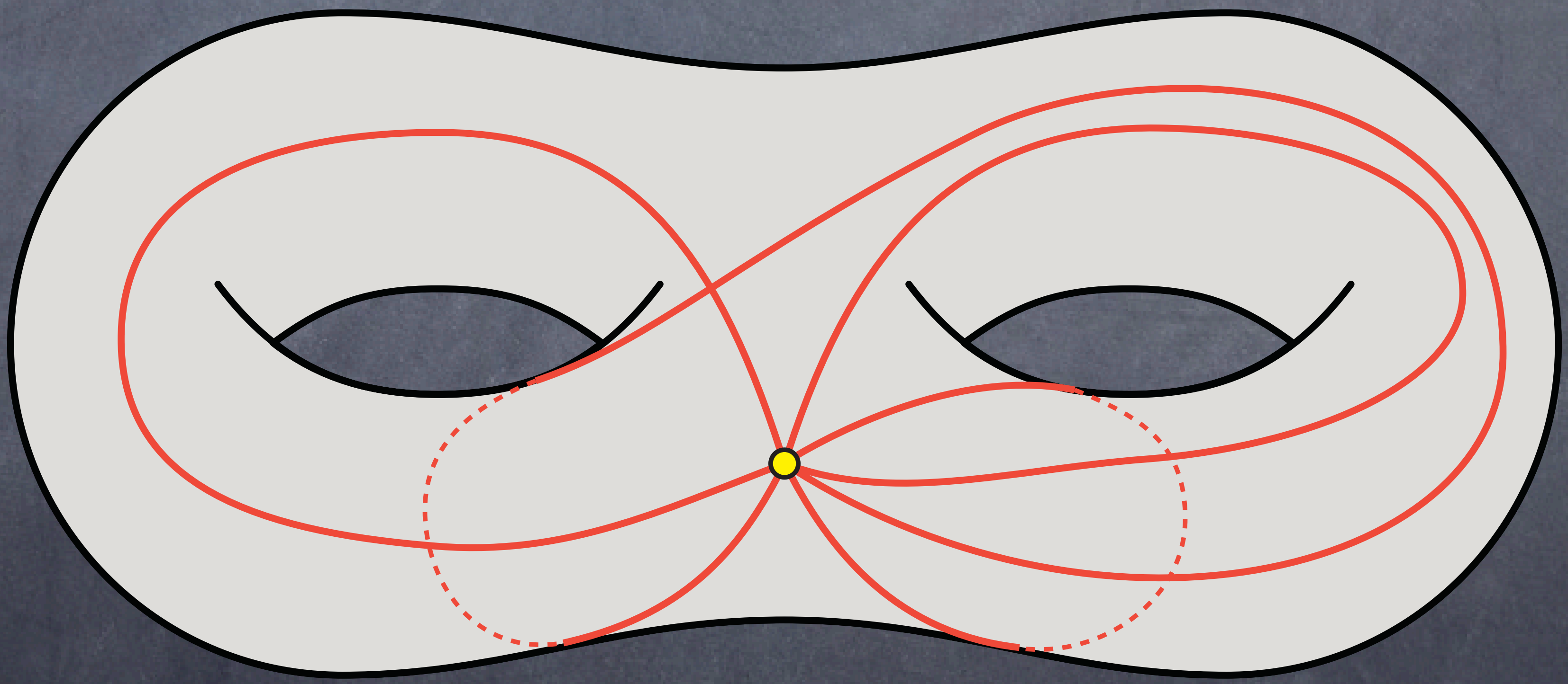
- $L \cdot L'(t)$:= if $t<1/2$ then $L(2t)$ else $L'(2t-1)$ fi
- Inverses: $\overline{L}(t) := L(1-t)$
- Identity: $1(t) := x$

The identity element is the homotopy class of contractible loops.



Homotopy basis

2g loops $\alpha_1, \alpha_2, \ldots, \alpha_{2g}$ that generate $\pi_1(M,x)$:
Every loop is homotopic to a concatenation of basis loops and/or their inverses.



Every system of loops is a homotopy basis,
but not vice versa!

The greedy homotopy basis

For each i, the greedy loop γ_i is
the shortest loop L such that
$M \backslash (\gamma_1 \cup \cdots \cup \gamma_{i-1} \cup L)$ is connected.

The set $\{\gamma_1, \gamma_2, \ldots, \gamma_{2g}\}$ is a system of loops,
so it is also a homotopy basis.

Proof of optimality

Shortest loops

Let $\sigma(uv)$ denote the shortest loop in G that contains the edge uv; this loop consists of

1. the shortest path from x to u,
2. the edge uv, and
3. the shortest path from v to x.

Lemma 1

Let T be the tree of shortest paths from x to every other vertex of G. Every greedy loop γ_i has the form $\sigma(e)$ for some edge $e \in G \backslash T$.

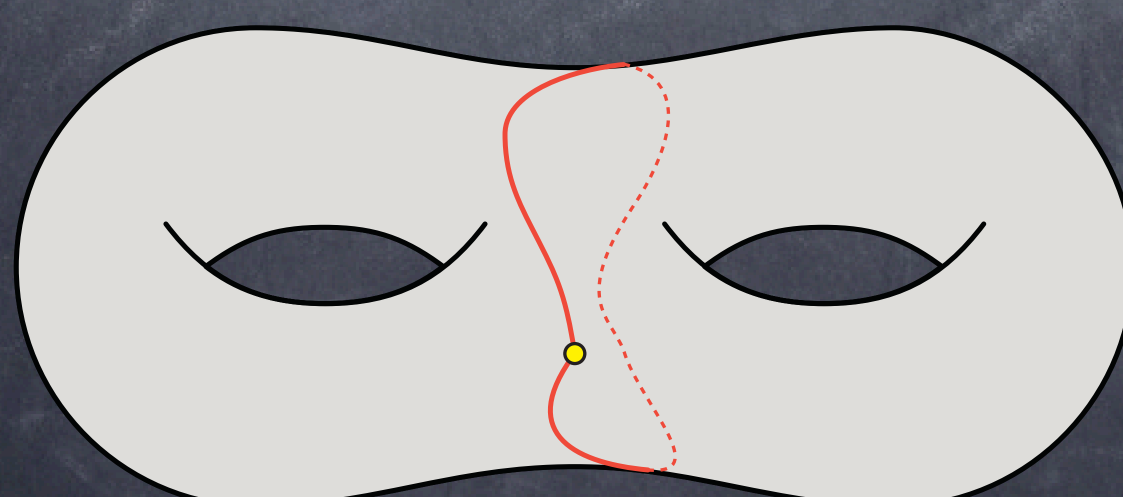
⬇

For each i, γ_i is the shortest loop $\sigma(e)$, over all edges $e \in G \backslash T$, such that $M \backslash (\gamma_1 \cup \cdots \cup \gamma_{i-1} \cup \sigma(e))$ is connected.

Homology

Any loop L has a fixed vector [L] of exponent sums with respect to any homotopy basis.

- For example, if $L \simeq \gamma_1\bar{\gamma}_2\gamma_3\bar{\gamma}_1\gamma_3\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_2\bar{\gamma}_2\gamma_1$ then $[L] = (0, -4, 2, 0)$.
- Linearity: $[L \cdot L'] = [L] + [L']$ and $[L] = -[L]$
- $[L] = (0,0,\ldots,0)$ iff L is a separating loop.



Greedy factors

We call γ_i a greedy factor of loop L if the corresponding exponent sum in [L] is non-zero.

- Each greedy loop is its only greedy factor.
- Separating loops have no greedy factors.

Lemma 2

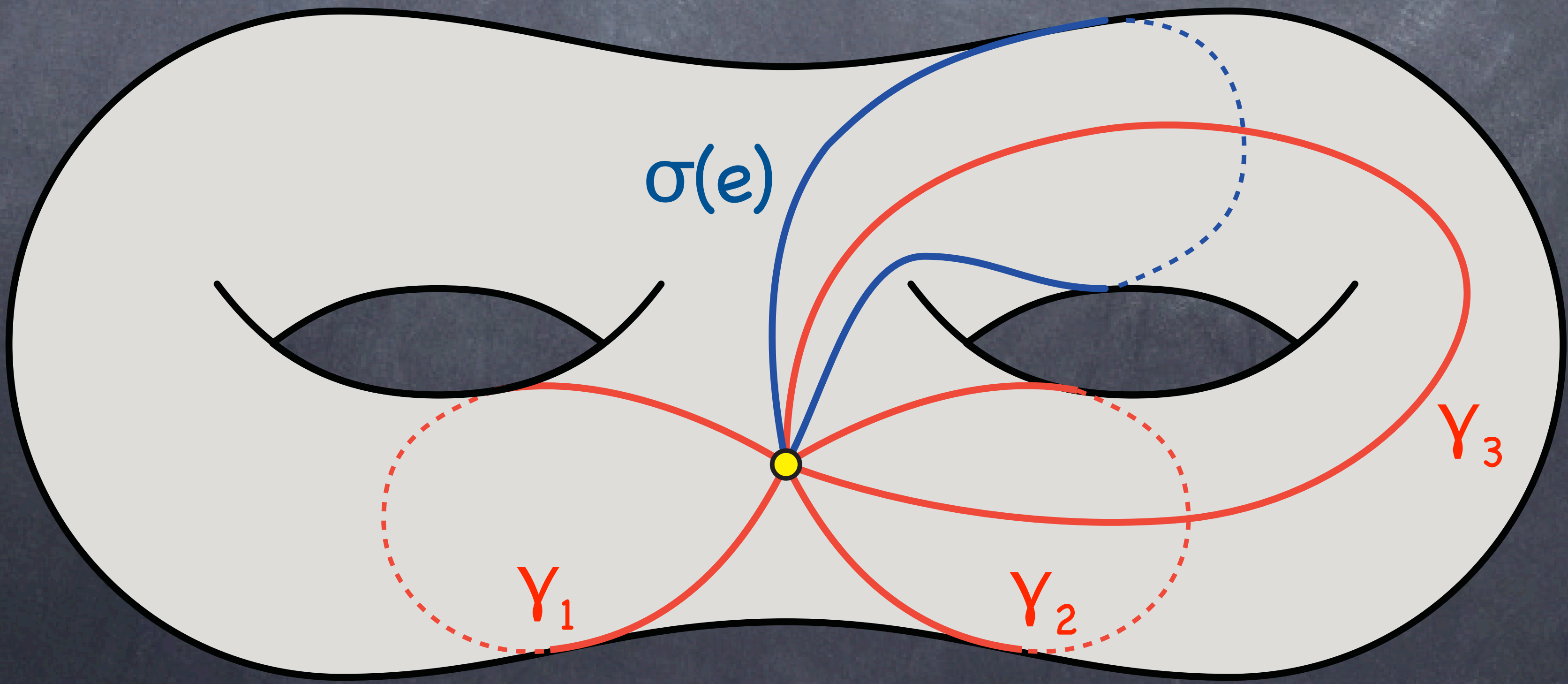
If γ is a greedy factor of $\sigma(e)$ for some edge $e \in G\backslash T$, then $|\gamma| \leq |\sigma(e)|$.

- If $\sigma(e)$ is a greedy loop, then $\gamma=\sigma(e)$.

Lemma 2

If γ is a greedy factor of $\sigma(e)$ for some edge $e \in G \setminus T$, then $|\gamma| \leq |\sigma(e)|$.

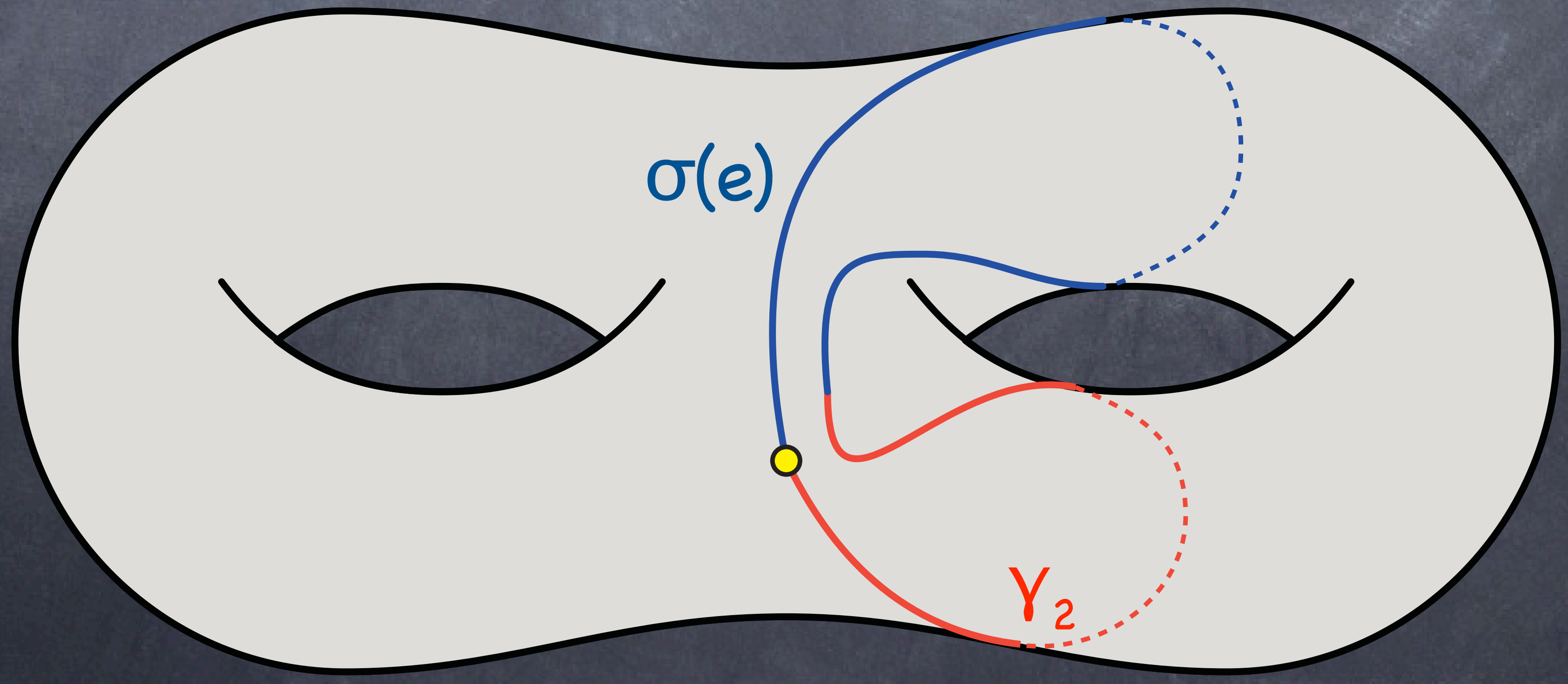
- Let $\gamma_1, \gamma_2, \ldots, \gamma_i$ be the greedy loops shorter than $\sigma(e)$.
- By definition, $M \setminus (\gamma_1 \cup \cdots \cup \gamma_i \cup \sigma(e))$ is disconnected.



Lemma 2

If γ is a greedy factor of $\sigma(e)$ for some edge $e \in G\backslash T$, then $|\gamma| \leq |\sigma(e)|$.

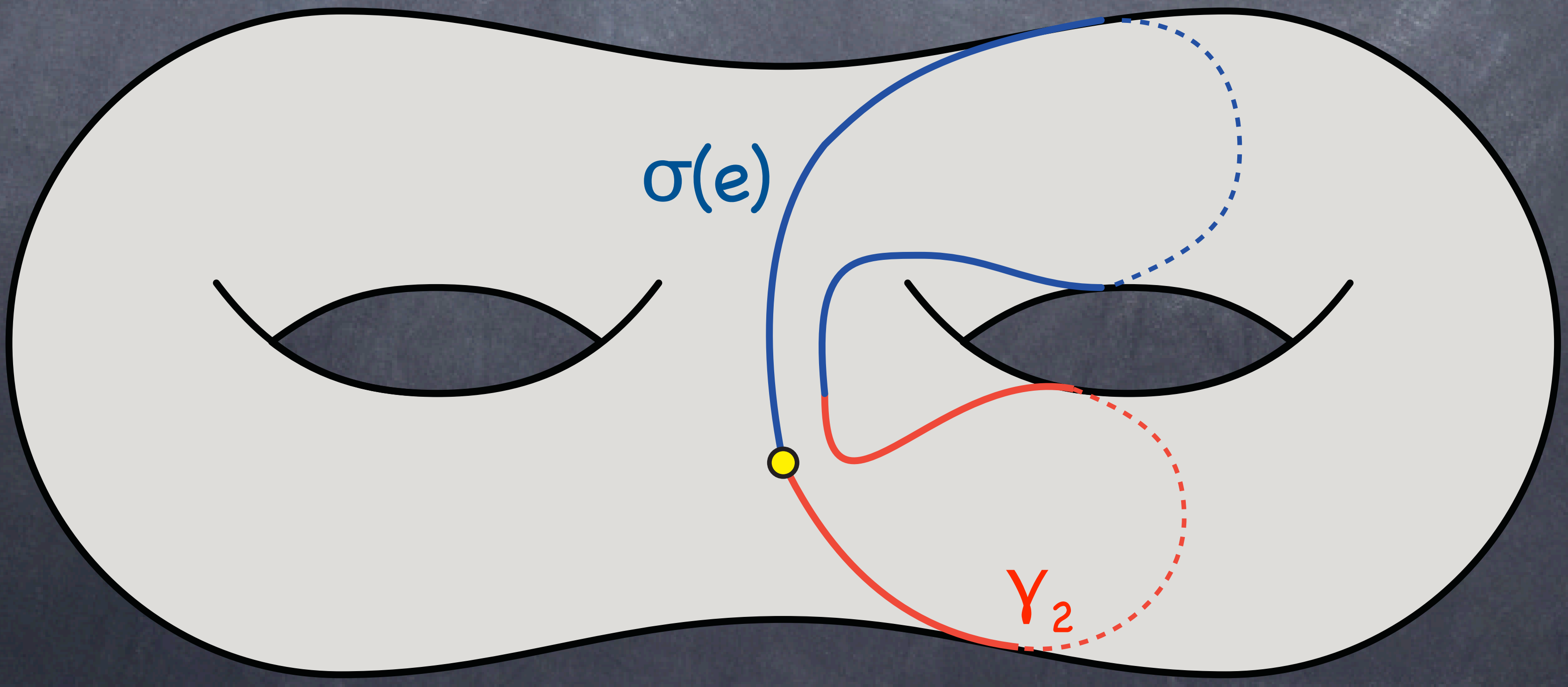
- Choose a minimal subset $\{\gamma_{j1}, \gamma_{j2}, \dots \gamma_{jr}\} \subseteq \{\gamma_1, \gamma_2, \dots, \gamma_i\}$ such that $M\backslash(\gamma_{j1} \cup \cdots \cup \gamma_{jk} \cup \sigma(e))$ is disconnected.
- $\sigma(e)\ \gamma_{j1}\cdots\gamma_{jr}$ is a separating loop.



Lemma 2

If γ is a greedy factor of $\sigma(e)$ for some edge $e \in G\backslash T$, then $|\gamma| \leq |\sigma(e)|$.

- $[\sigma(e)\ \gamma_{j1} \cdots \gamma_{jr}] = (0,0,\ldots 0)$, so $[\sigma(e)] = -[\gamma_{j1}] - \cdots - [\gamma_{jr}]$.
- Thus $\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{jr}$ are the only greedy factors of $\sigma(e)$.



Lemma 3

If γ is a greedy factor of **any** loop L, then $|\gamma| \leq |L|$.

- Let $e_1, e_2, \ldots, e_r$ be the sequence of edges in $G \setminus T$ traversed by L.
- $L \simeq \sigma(e_1) \cdot \sigma(e_2) \cdots \sigma(e_r)$.
- If γ is a greedy factor of L, then for some i, γ is a greedy factor of $\sigma(e_i)$, so $|\gamma| \leq |\sigma(e_i)|$.
- By definition, $|\sigma(e_i)| \leq |L|$ for all i.

Lemma 4

Any homotopy basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}$ can be reordered so that γ_i is a greedy factor of α_i for all i.

- There is a nonsingular linear transformation mapping $[L]_\alpha$ to $[L]_\gamma$.
- The ith column of the transformation matrix is the vector $[\alpha_i]_\gamma$.
- For some perm. π, $\mathbf{u}_{\mathbf{i},\pi(\mathbf{i})} \neq \mathbf{0}$ for all i.

Main Theorem

Any homotopy basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}$

can be reordered so that

$|\gamma_i| \leq |\alpha_i|$ for all i.

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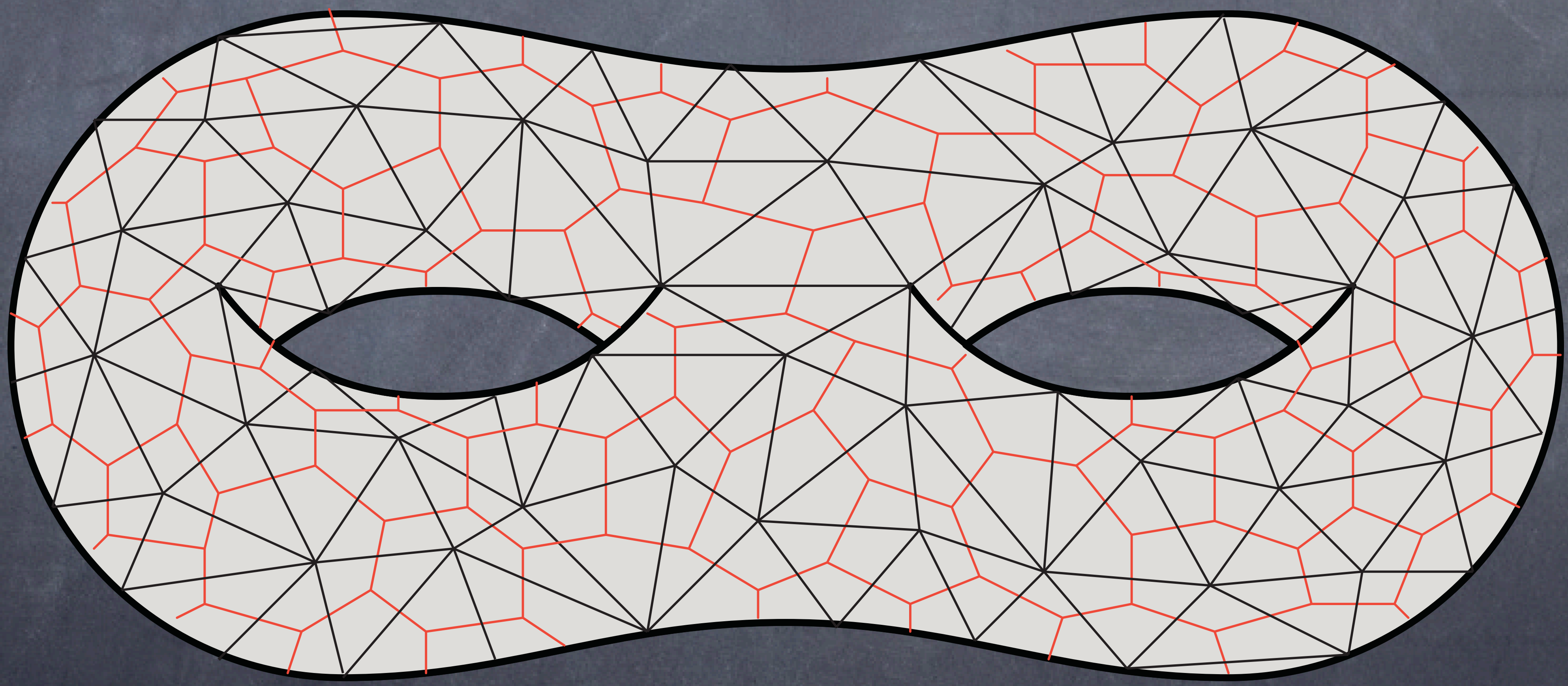
No system of loops has smaller total length than the greedy homotopy basis.

The Algorithm

Dual graph G*

Each edge e in G has a dual edge e* in G*.



Tree/co-tree decomposition

[Eppstein 2004]

- Let T = any spanning tree of G.
- Let T^* = any spanning tree of $(G \setminus T)^*$.
- Then $\{\sigma(e) \mid e \notin T \text{ and } e^* \notin T^*\}$ is a system of loops!

Our greedy algorithm

- Let T = shortest path tree rooted at x.
  [$O(n \log n)$ time with Dijkstra's algorithm]

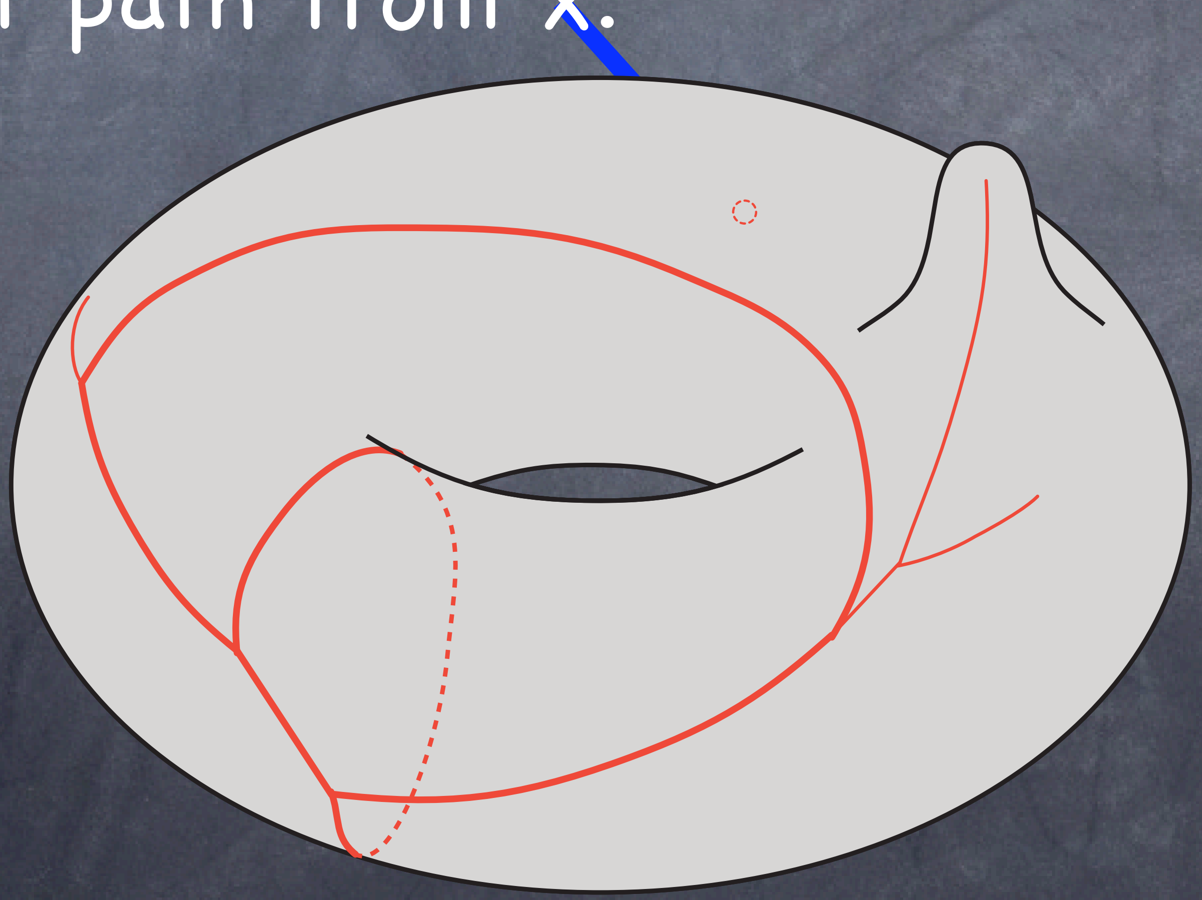
- Let T^* = maximum spanning tree of $(G \backslash T)^*$ where $\text{weight}(e^*) = |\sigma(e)|$.
  [$O(n \log n)$ time with any textbook MST algorithm]

- Then $\{\sigma(e) \mid e \notin T \text{ and } e^* \notin T^*\}$ is the greedy homotopy basis!

Extensions

Other surface types

- Our greedy characterization is also valid for smooth and piecewise-linear surfaces.
- Cut locus = points with more than one shortest path from x.



Other surface types

- For each arc ϕ of the cut locus, let $\sigma(\phi)$ denote the shortest loop crossing ϕ.
- Lemma 1: Every greedy loop is $\sigma(\phi)$ for some arc ϕ of the cut locus.
- The rest of the optimality proof is identical.

Other surface types

- Moreover, every greedy loop is $\sigma(\phi)$ for some ϕ not in the maximum spanning tree of the cut locus, where weight$(\phi) = |\sigma(\phi)|$.
- Once we compute the cut locus, the rest of the algorithm takes $O(g \log g)$ time.
- PL surfaces: $O(n^2)$ time [Chen and Han'96]

Thank you!