Greedy Optimal Homotopy and Homology Generators

Jeff Erickson and Kim Whittlesey
University of Illinois, Urbana-Champaign

To appear at SODA 2005
What's the problem?

Given a topologically complex surface, cut it into one or more topological disks.

- Simplification, remeshing, compression, texture mapping, parameterization, . . .
- Computing separators and tree decompositions of non-planar graphs

Moreover, cut the surface as little as possible.
An embedded graph whose removal cuts the surface into a single topological disk.
Optimality is hard

Erickson, Har-Peled 2000:

- Computing the minimum-length cut graph for a given surface is \textit{NP}-hard.

- A \textit{O}(\log^2 g)\text{-approximation} can be found in \textit{O}(g^2n \log n) time, where \( g \) is the genus of the surface.
System of loops

A cut graph with one vertex, or equivalently, a set of $2g$ loops through a common basepoint
Optimality is not so hard?

Colin de Verdière, Lazarus 2002:

- Given a system of loops on a surface, the shortest system of loops in the same homotopy class can be computed in (large) polynomial time.*

*under some mild (and probably unnecessary) assumptions.
Optimality is easy!

Our main result:

- The shortest system of loops with a given basepoint can be computed in $O(n \log n)$ time using a simple greedy algorithm.

- The overall shortest system of loops can be computed in $O(n^2 \log n)$ time.

- More machinery improves both running times by a factor of $O(\log n)$ if $g=O(n^{0.999})$. 
Outline

- **Definitions:** combinatorial surface, homotopy, homotopy basis, exponent sums, greedy homotopy basis, homology

- **Proof of optimality**

- **Implementation:** dual graph, shortest paths + max spanning tree
Surfaces and topology
Combinatorial surface

An abstract compact orientable 2-manifold with a weighted graph $G$, embedded so that every face is a disk.
Loop

The image of a function $L:[0,1] \rightarrow G$ such that $L(0) = L(1) = x$, where $x$ is a fixed basepoint.

Our algorithms consider only circuits in $G$. 
Homotopy

Two loops $L$ and $L'$ are homotopic (written $L \simeq L'$) if one loop can be continuously deformed into the other.

Formally, a homotopy from $L$ to $L'$ is a continuous function $h : [0,1] \times [0,1] \rightarrow M$ where $h(0,t) = L(t)$, $h(1,t) = L'(t)$, and $h(s,0) = h(s,1) = x$. 
The fundamental group

The set of homotopy classes form a group \( \pi_1(M,x) \) under concatenation:

- \( L \cdot L'(t) := \text{if } t<1/2 \text{ then } L(2t) \text{ else } L'(2t-1) \text{ fi} \)
- Inverses: \( \overline{L}(t) := L(1-t) \)
- Identity: \( 1(t) := x \)

The identity element is the homotopy class of contractible loops.
Homotopy basis

2g loops $\alpha_1, \alpha_2, \ldots, \alpha_{2g}$ that generate $\pi_1(M,x)$:
Every loop is homotopic to a concatenation of basis loops and/or their inverses.

Every system of loops is a homotopy basis, but not vice versa!
The greedy homotopy basis

For each $i$, the greedy loop $\gamma_i$ is the shortest loop $L$ such that $M \setminus (\gamma_1 \cup \cdots \cup \gamma_{i-1} \cup L)$ is connected.

The set $\{\gamma_1, \gamma_2, \ldots, \gamma_{2g}\}$ is a system of loops, so it is also a homotopy basis.
Proof of optimality
Shortest loops

Let $\sigma(\mathit{uv})$ denote the shortest loop in $G$ that contains the edge $\mathit{uv}$; this loop consists of

1. the shortest path from $x$ to $u$,
2. the edge $\mathit{uv}$, and
3. the shortest path from $v$ to $x$.  

Lemma 1

Let $T$ be the tree of shortest paths from $x$ to every other vertex of $G$. Every greedy loop $\gamma_i$ has the form $\sigma(e)$ for some edge $e \in G \setminus T$.

For each $i$, $\gamma_i$ is the shortest loop $\sigma(e)$, over all edges $e \in G \setminus T$, such that $M \setminus (\gamma_1 \cup \cdots \cup \gamma_{i-1} \cup \sigma(e))$ is connected.
Any loop $L$ has a fixed vector $[L]$ of exponent sums with respect to any homotopy basis.

For example, if $L = \gamma_1 \bar{\gamma}_2 \gamma_3 \bar{\gamma}_1 \gamma_3 \bar{\gamma}_1 \gamma_2 \bar{\gamma}_2 \gamma_1$ then $[L] = (0, -4, 2, 0)$.

Linearity: $[L \cdot L'] = [L] + [L']$ and $[L] = -[L]$.

$[L] = (0,0,...,0)$ iff $L$ is a separating loop.
Greedy factors

We call $\gamma_i$ a greedy factor of loop $L$ if the corresponding exponent sum in $[L]$ is non-zero.

- Each greedy loop is its only greedy factor.
- Separating loops have no greedy factors.
Lemma 2

If $\gamma$ is a greedy factor of $\sigma(e)$ for some edge $e \in G \setminus T$, then $|\gamma| \leq |\sigma(e)|$.

If $\sigma(e)$ is a greedy loop, then $\gamma = \sigma(e)$. 
Lemma 2

If γ is a greedy factor of σ(e) for some edge e ∈ G\T, then |γ| ≤ |σ(e)|.

Let γ₁, γ₂, ..., γᵢ be the greedy loops shorter than σ(e).

By definition, $M(γ₁ ∪ ... ∪ γᵢ ∪ σ(e))$ is disconnected.
Lemma 2

If $\gamma$ is a greedy factor of $\sigma(e)$ for some edge $e \in G \setminus T$, then $|\gamma| \leq |\sigma(e)|$.

- Choose a minimal subset $\{Y_{j1}, Y_{j2}, \ldots, Y_{jr}\} \subseteq \{Y_1, Y_2, \ldots, Y_i\}$ such that $M \setminus (Y_{j1} \cup \cdots \cup Y_{jk} \cup \sigma(e))$ is disconnected.

- $\sigma(e) \gamma_{j1} \cdots \gamma_{jr}$ is a separating loop.
Lemma 2

If $\gamma$ is a greedy factor of $\sigma(e)$ for some edge $e \in G \setminus T$, then $|\gamma| \leq |\sigma(e)|$.

- $[\sigma(e) \gamma_1 \ldots \gamma_{jr}] = (0,0,\ldots,0)$, so $[\sigma(e)] = -[\gamma_1] - \cdots - [\gamma_{jr}]$.
- Thus $\gamma_1, \gamma_2, \ldots, \gamma_{jr}$ are the only greedy factors of $\sigma(e)$. 
Lemma 3

If γ is a greedy factor of any loop L, then |γ| ≤ |L|.

Let $e_1, e_2, \ldots, e_r$ be the sequence of edges in $G \backslash T$ traversed by L.

$L ≃ σ(e_1)·σ(e_2)\cdotsσ(e_r)$.

If γ is a greedy factor of L, then for some i, γ is a greedy factor of $σ(e_i)$, so |γ| ≤ |σ(e_i)|.

By definition, |σ(e_i)| ≤ |L| for all i.
Lemma 4

Any homotopy basis \(\{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}\) can be reordered so that \(\gamma_i\) is a greedy factor of \(\alpha_i\) for all \(i\).

- There is a nonsingular linear transformation mapping \([L]_\alpha\) to \([L]_\gamma\).
- The \(i\)th column of the transformation matrix is the vector \([\alpha_i]_\gamma\).
- For some perm. \(\pi\), \(u_{i,\pi(i)} \neq 0\) for all \(i\).
Main Theorem

Any homotopy basis \( \{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\} \) can be reordered so that \( |\gamma_i| \leq |\alpha_i| \) for all \( i \).

No system of loops has smaller total length than the greedy homotopy basis.
The Algorithm
Dual graph $G^*$

Each edge $e$ in $G$ has a dual edge $e^*$ in $G^*$. 
Tree/co-tree decomposition

[Eppstein 2004]

Let $T = \text{any spanning tree of } G$.

Let $T^* = \text{any spanning tree of } (G\setminus T)^*$.

Then $\{\sigma(e) \mid e \in T \text{ and } e^* \notin T^*\}$ is a system of loops!
Our greedy algorithm

Let $T = \text{shortest path tree rooted at } x$. 
$[O(n \log n) \text{ time with Dijkstra's algorithm}]$

Let $T^* = \text{maximum spanning tree of } (G \setminus T)^*$
where $\text{weight}(e^*) = |\sigma(e)|$. 
$[O(n \log n) \text{ time with any textbook MST algorithm}]$

Then $\{\sigma(e) \mid e \in T \text{ and } e^* \in T^*\}$ is the greedy homotopy basis!
Extensions
Other surface types

Our greedy characterization is also valid for smooth and piecewise-linear surfaces.

Cut locus = points with more than one shortest path from x.
Other surface types

For each arc $\phi$ of the cut locus, let $\sigma(\phi)$ denote the shortest loop crossing $\phi$.

**Lemma 1:** Every greedy loop is $\sigma(\phi)$ for some arc $\phi$ of the cut locus.

The rest of the optimality proof is identical.
Other surface types

Moreover, every greedy loop is $\sigma(\phi)$ for some $\phi$ not in the maximum spanning tree of the cut locus, where weight($\phi$) = $|\sigma(\phi)|$.

Once we compute the cut locus, the rest of the algorithm takes $O(g \log g)$ time.

PL surfaces: $O(n^2)$ time [Chen and Han`96]
Thank you!