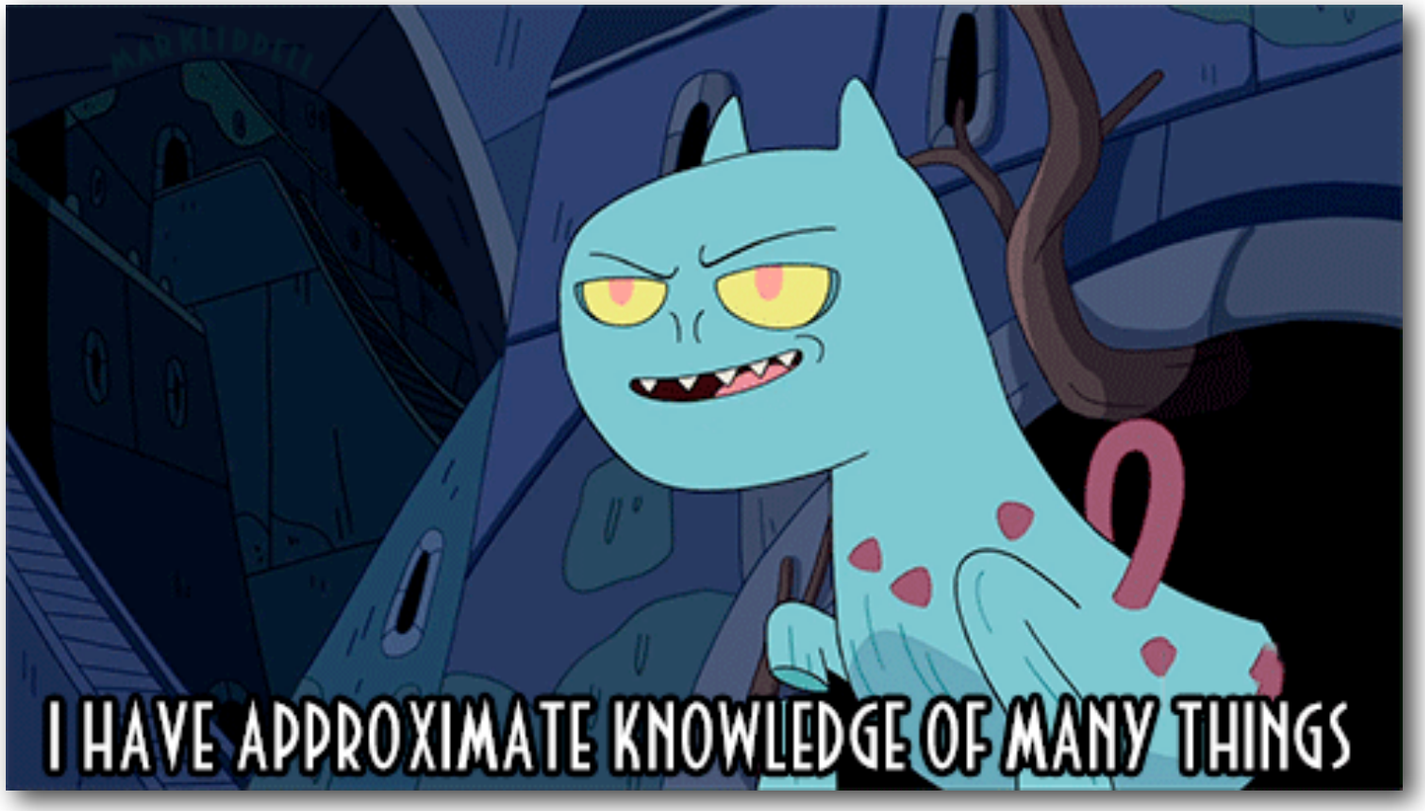
Theoretical Advances in Hexahedral Mesh Generation

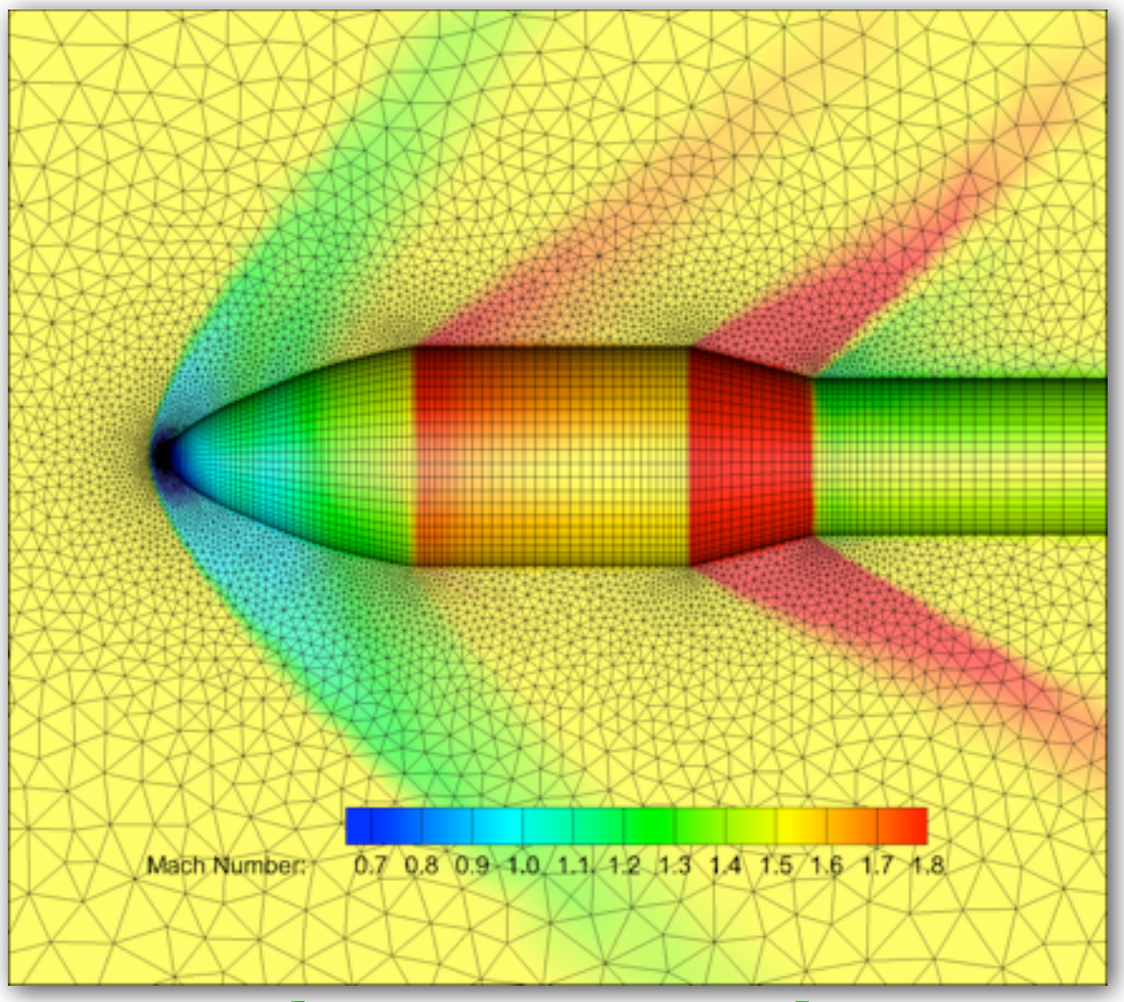
Jeff Erickson
University of Illinois, Urbana-Champaign


I don't know why anyone would
think I have any authority to
comment on that.


I HAVE APPROXIMATE KNOWLEDGE OF MANY THINGS

Why meshing?

- Numerical solution of differential equations via finite element methods
  - Decompose simulation volume into elementary pieces — “elements”
  - Approximate solution within each element as a low-degree polynomial.

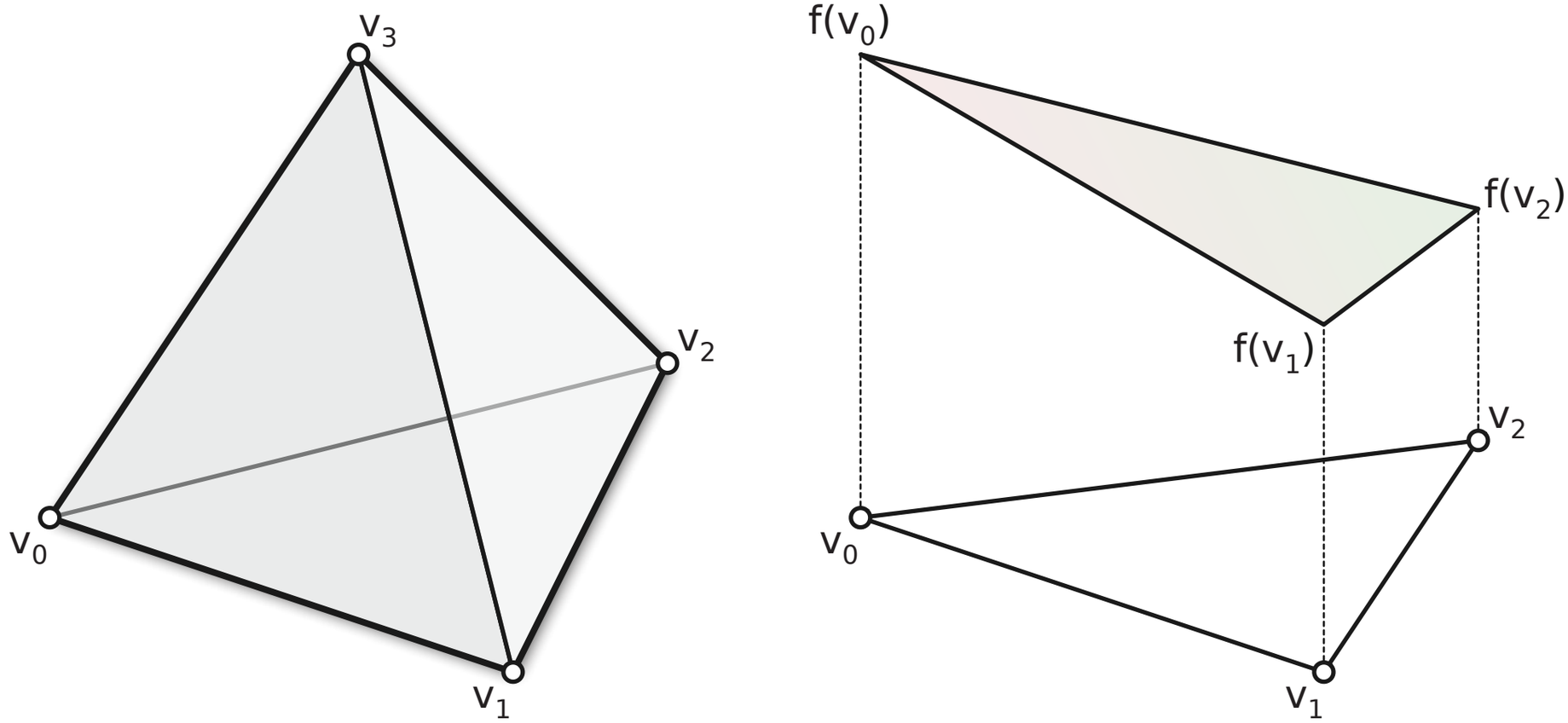


[blog.pointwise.com]

Typical finite elements

- Linear functions over tetrahedra

$$f(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \lambda_0 \cdot f(v_0) + \lambda_1 \cdot f(v_1) + \lambda_2 \cdot f(v_2) + \lambda_3 \cdot f(v_3)$$



- Higher-degree polynomials are also used

Typical finite elements

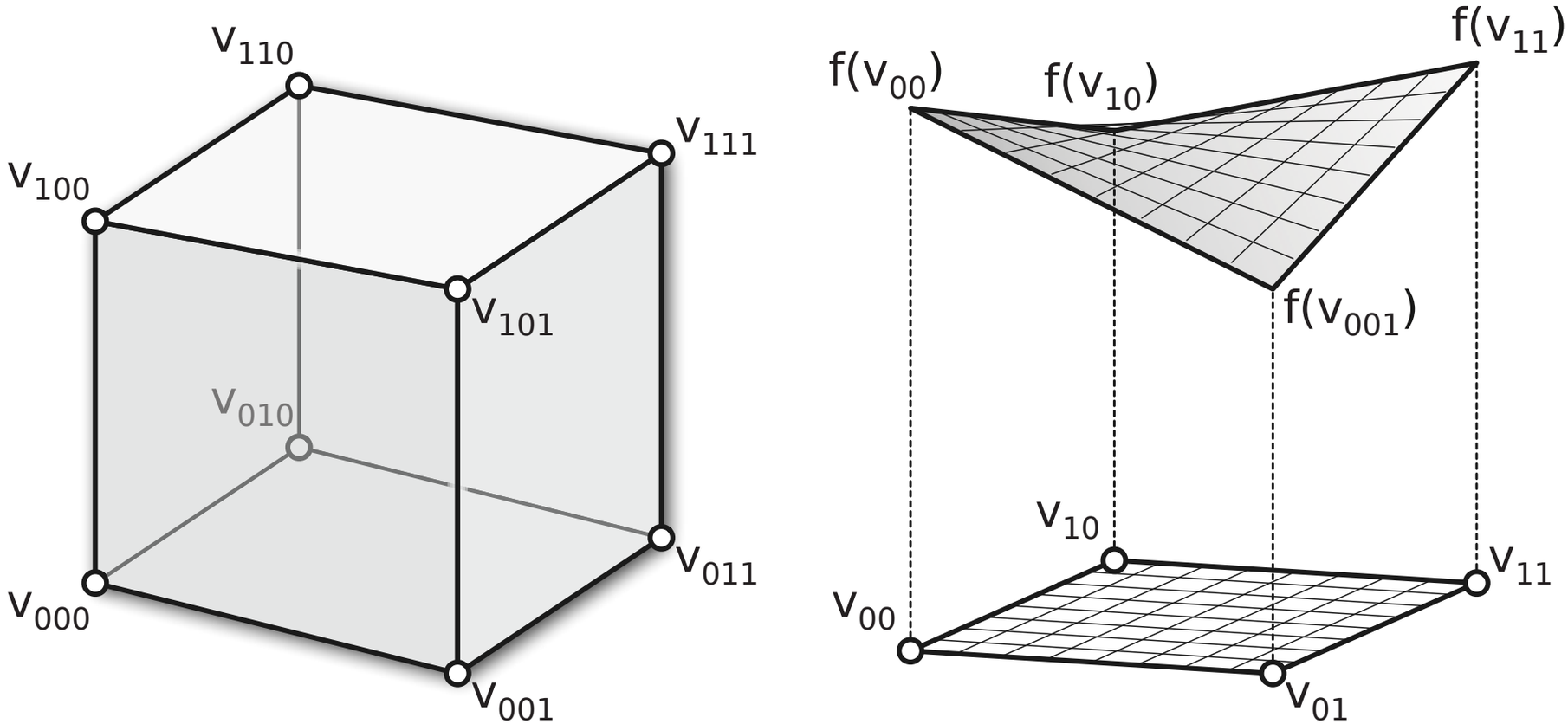
- Trilinear functions over hexahedra

$$
\begin{aligned}
f(\alpha_1, \alpha_2, \alpha_3) = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3) \cdot f(v_{000}) \\
(1-\alpha_1)(1-\alpha_2)\alpha_3 \cdot f(v_{001}) \\
(1-\alpha_1)\alpha_2(1-\alpha_3) \cdot f(v_{010}) \\
(1-\alpha_1)\alpha_2\alpha_3 \cdot f(v_{011}) \\
\alpha_1(1-\alpha_2)(1-\alpha_3) \cdot f(v_{100}) \\
\alpha_1(1-\alpha_2)\alpha_3 \cdot f(v_{101}) \\
\alpha_1\alpha_2(1-\alpha_3) \cdot f(v_{110}) \\
\alpha_1\alpha_2\alpha_3 \cdot f(v_{111})
\end{aligned}
$$

- Higher-degree polynomials and other element shapes are also used

Typical finite elements

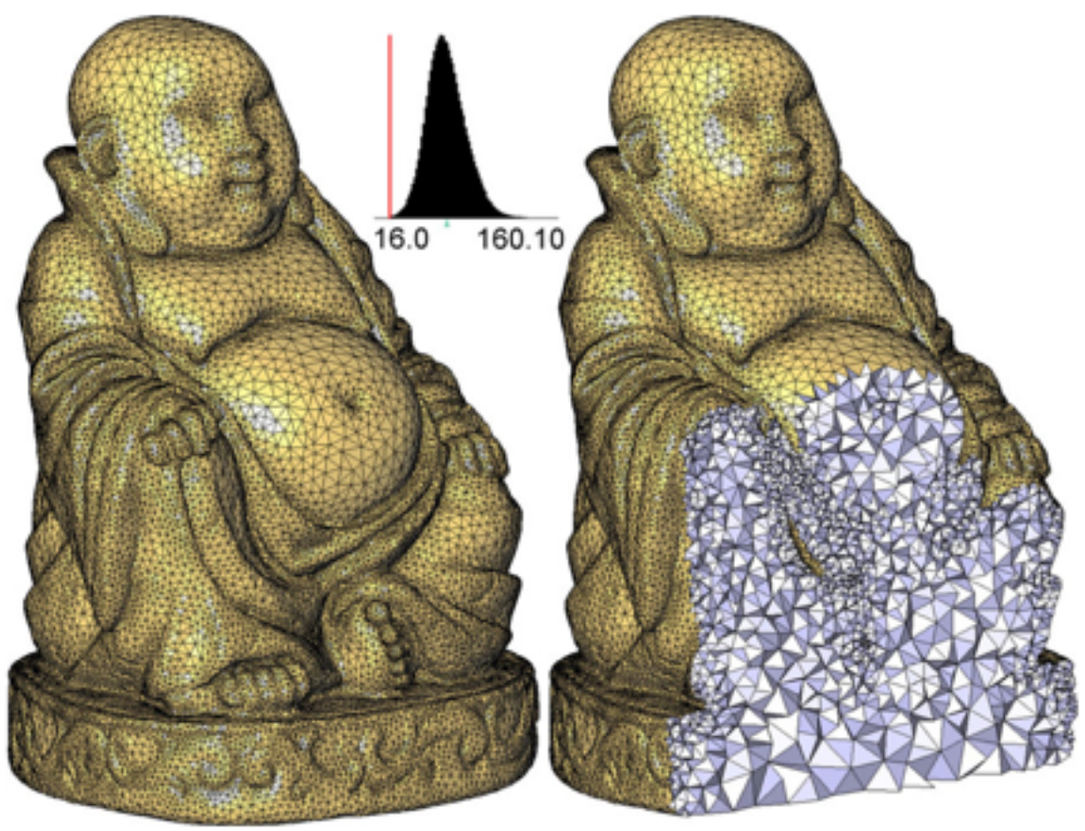
- Trilinear functions over hexahedra



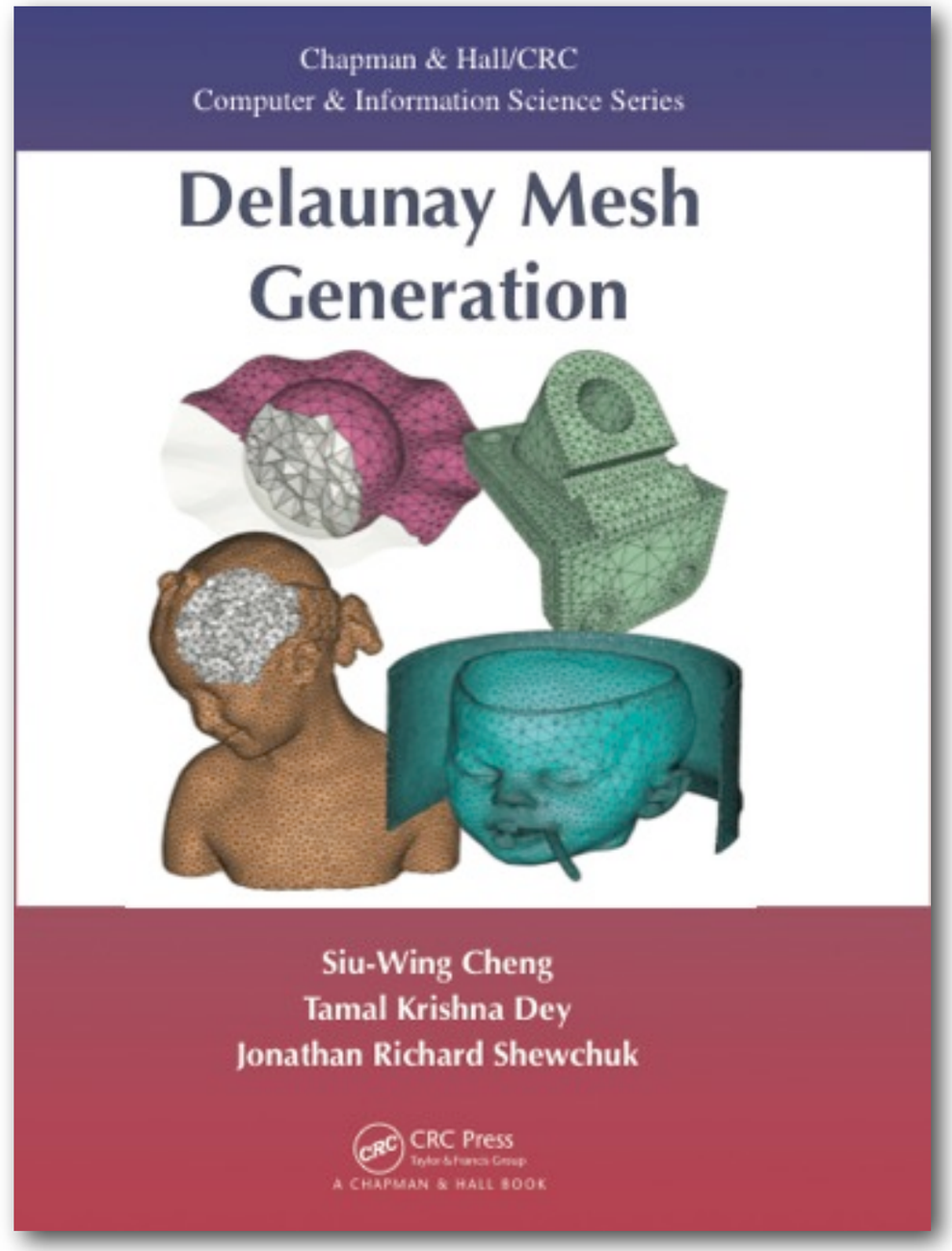
- Higher-degree polynomials and other element shapes are also used

Tetrahedral meshing

- Well-developed theory and robust general-purpose software

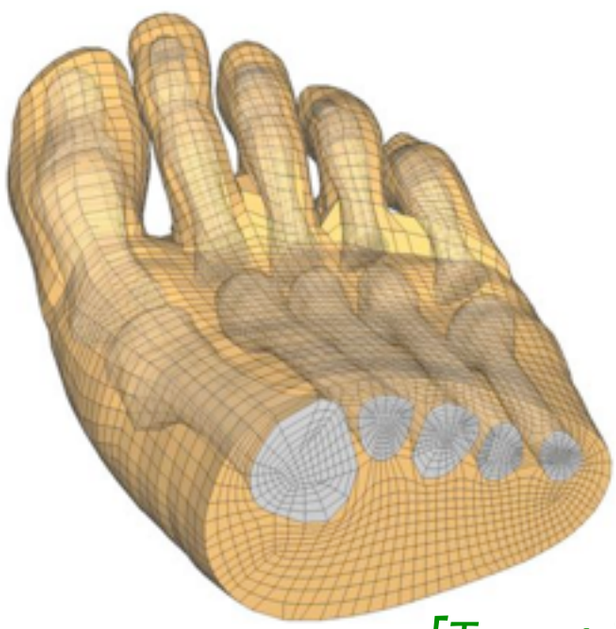


[Tournier Wormser Alliez Desbrun 2009]

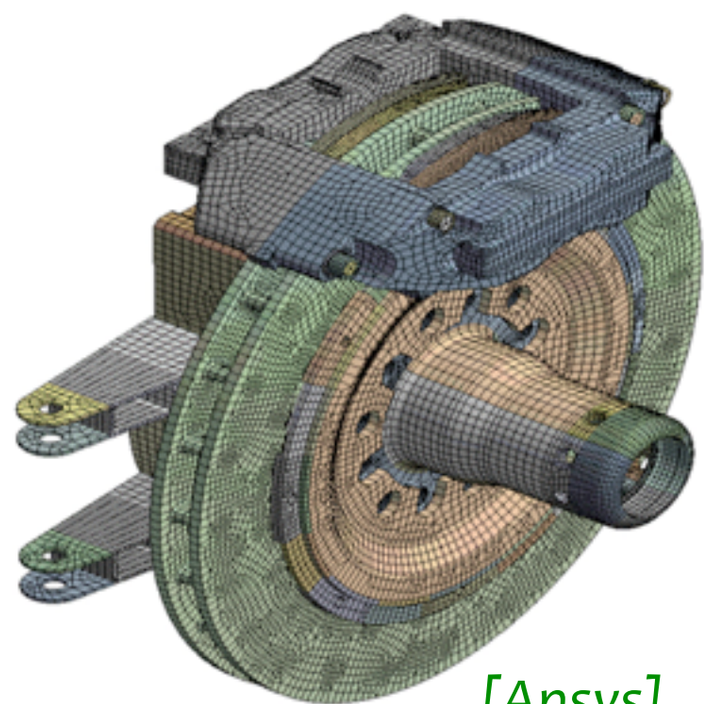


Hexahedral meshing

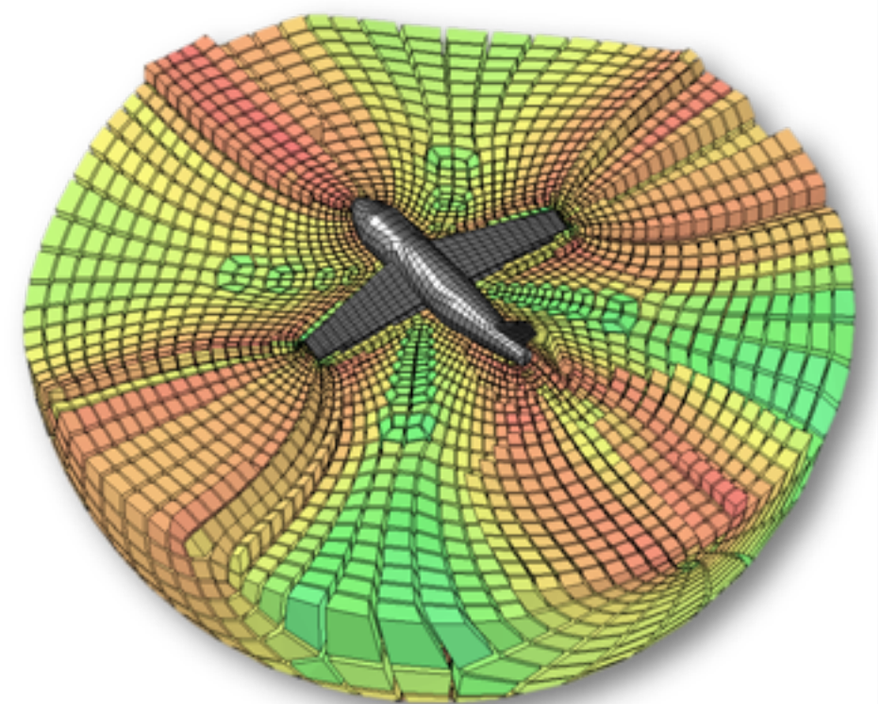
Huge variety of practical heuristics, engineering expertise, and software, but no automatic general methods. In other words, no ***algorithms***.



[Truegrid]



[Ansys]

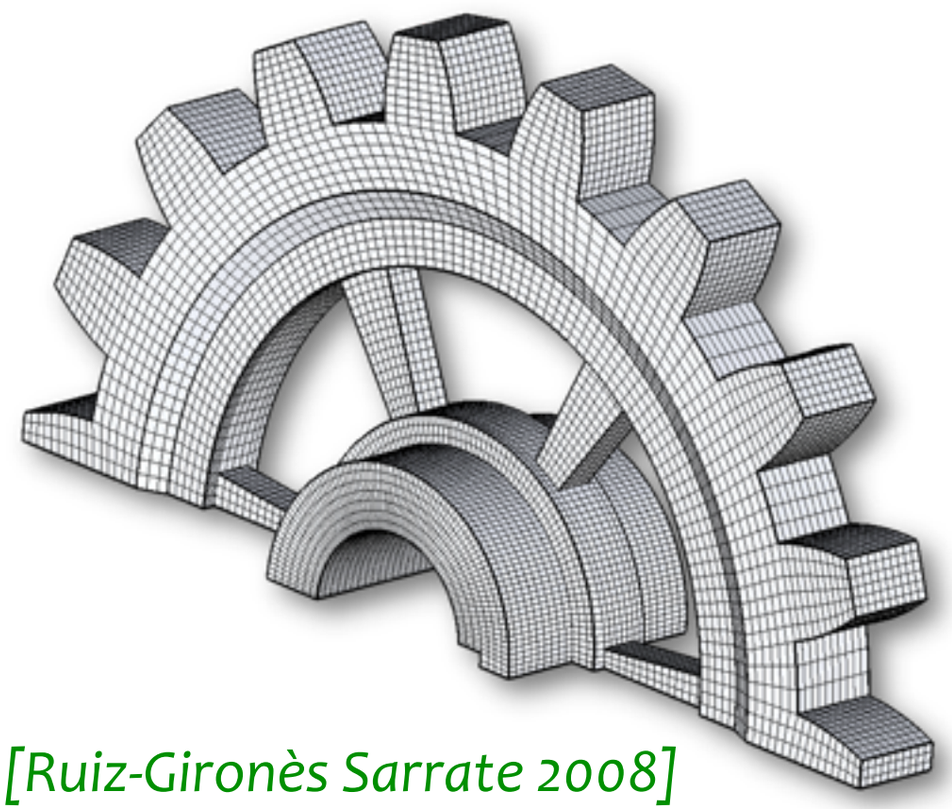


[Ruiz-Gironès Roca Sarrate 2012]

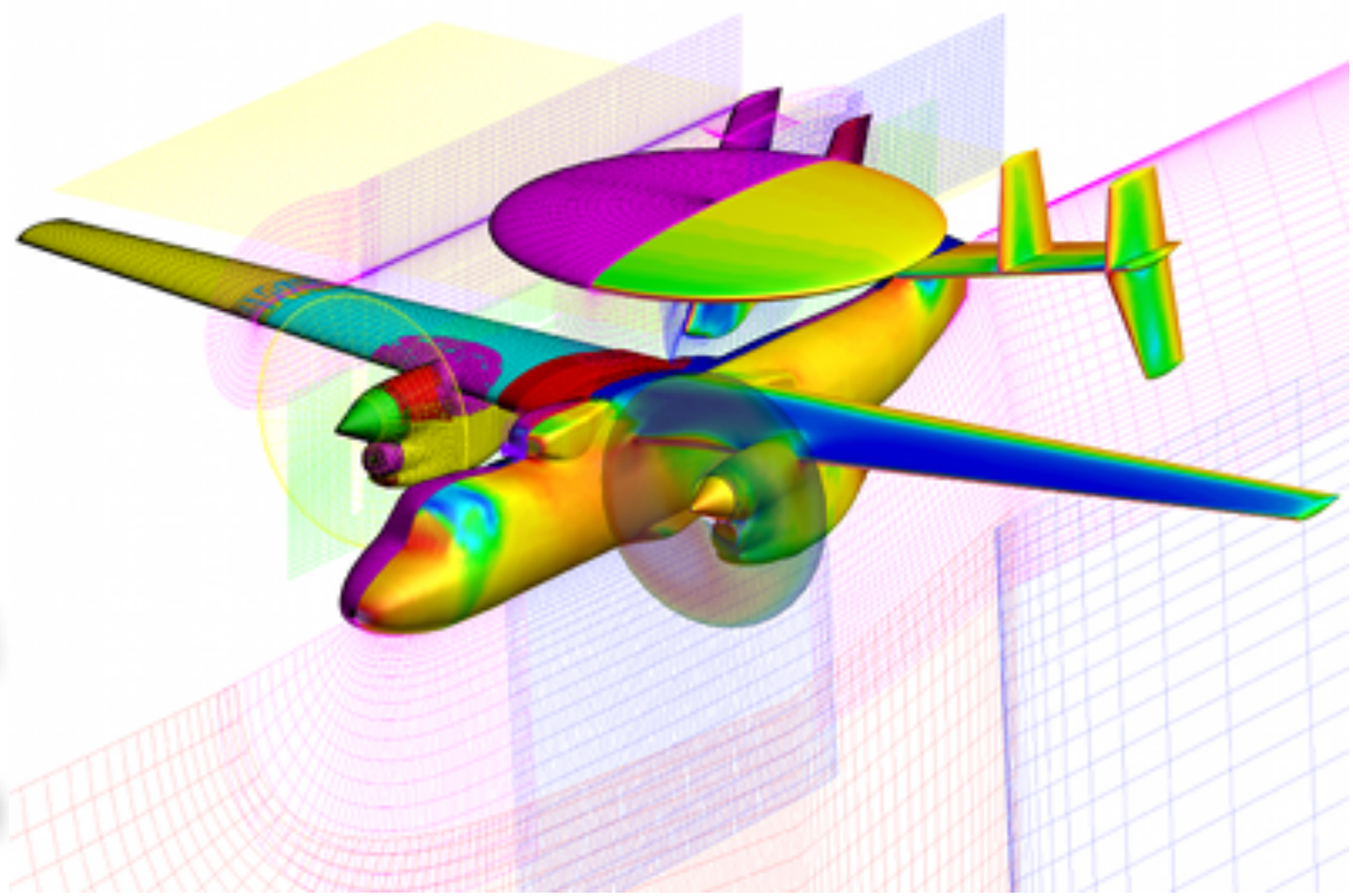
heu · ris · tic /*hyōō-ˈris-tik*/ [Gr. εὑρίσκω = find, discover]
n. An algorithm that doesn't work.

Why hex meshing?

- Hex meshes are better for some finite element methods, for some applications, and for some classes of geometry.
- But advantages are subtle!

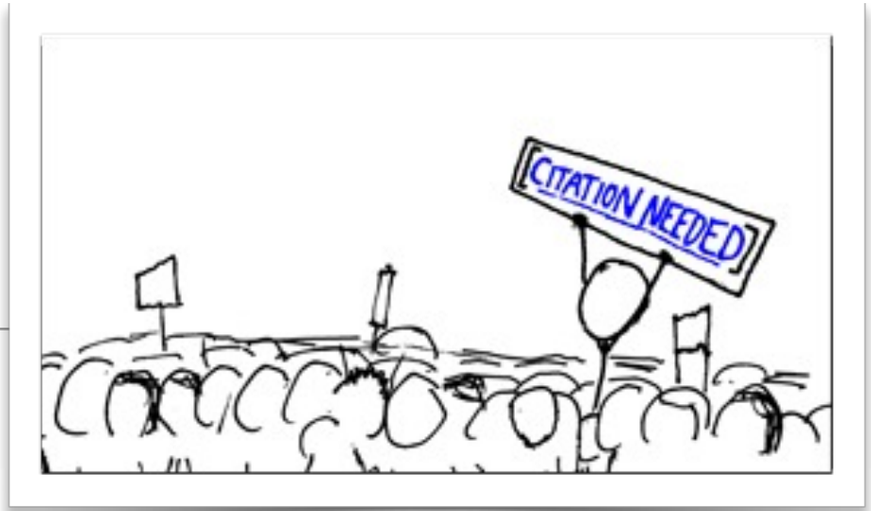


[Ruiz-Gironès Sarrate 2008]



[Gridgen]

Why hex meshing?

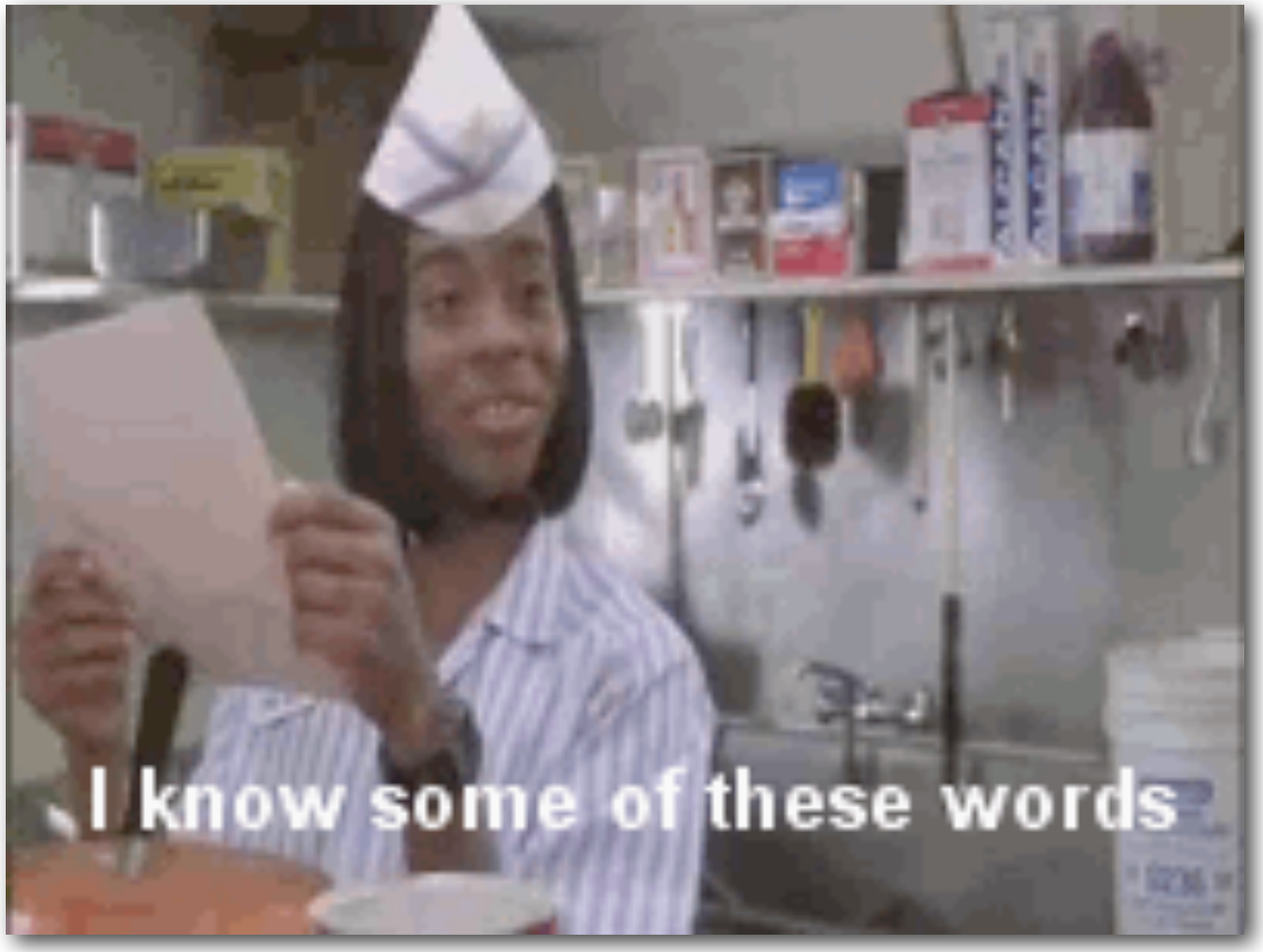


"Received wisdom" makes unfair comparisons:

- *Unstructured* tet meshes versus (locally) *structured* hex meshes
  - Fewer elements for a given number of nodes
    - *For regular cubical grids, sure, but for arbitrary point sets?*
    - *For many finite-element methods, the size of the linear system size depends on the number of **nodes**, not the number of **elements**.*
  - Exploit tensor product structure and/or anisotropy in solution
  - But structured meshes require special geometry

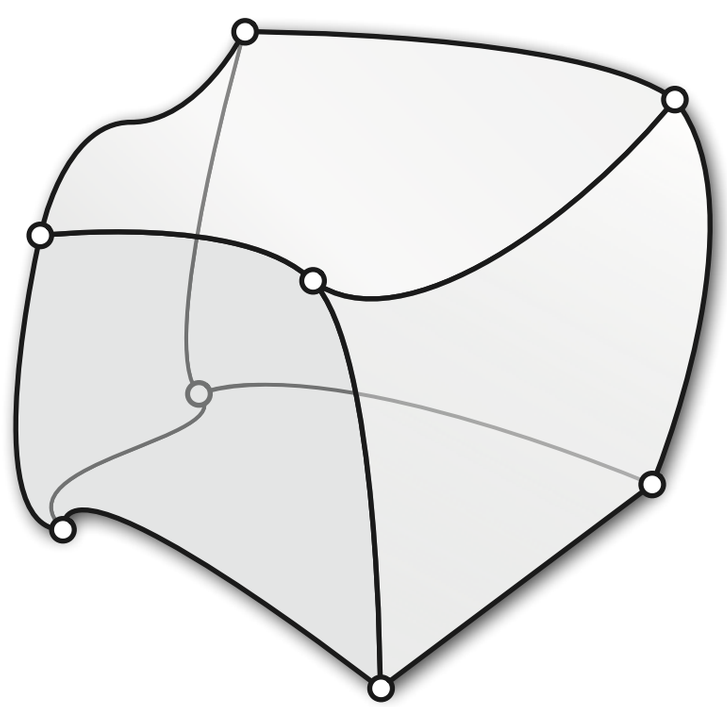
- *Linear* tet elements versus *multilinear* hex elements
  - Fewer elements for same accuracy
  - Avoid shear and volume locking
  - But higher-order tet elements avoid these problems! *[Weingarten 94]*

Defining our terms



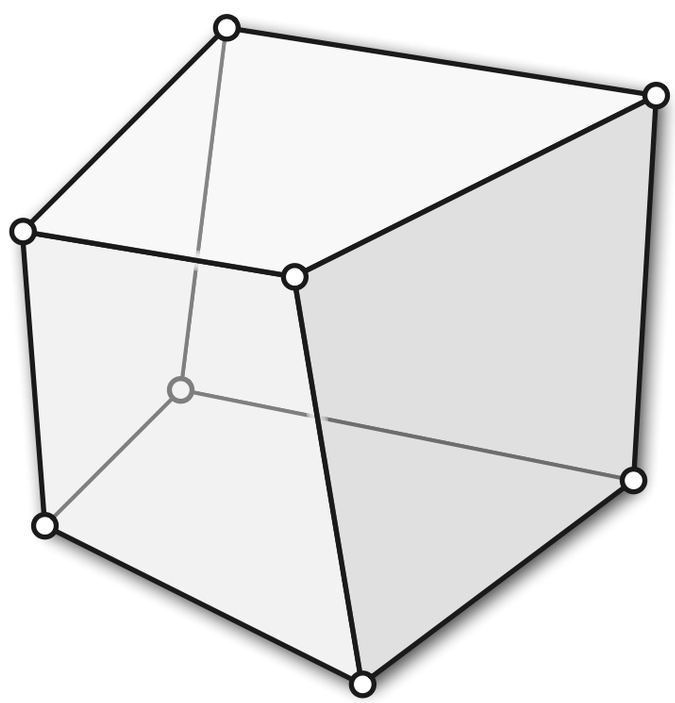
What's a "hexahedron"?

- **Topological:** What (most) mathematicians mean
  - A *topological cube* is the image of an injective map $q : [0,1]^3 \hookrightarrow \boldsymbol{R}^3$
  - Topological ball with boundary subdivision: 8 vertices, 12 edges, 6 facets



What's a "hexahedron"?

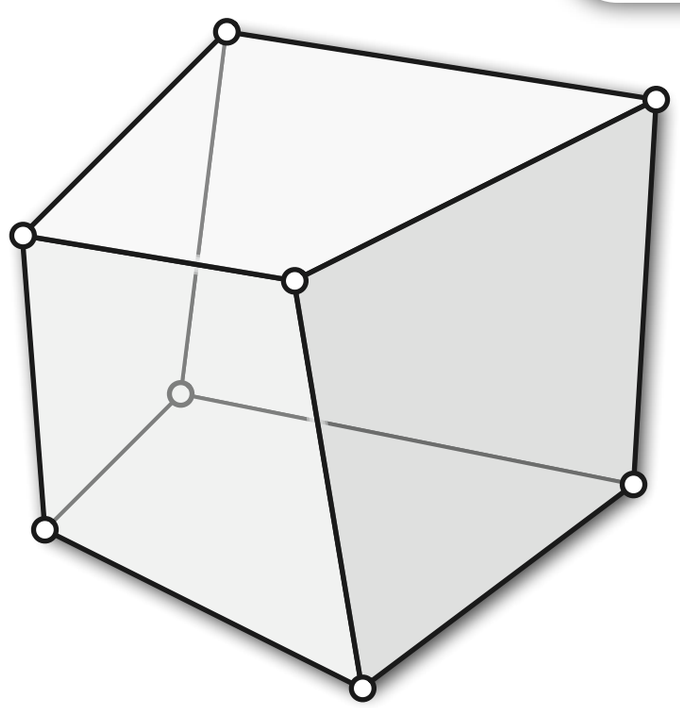
- **Polyhedral:** What computational geometers mean
  - A *hexahedron* is a convex polytope isomorphic to the cube $[0,1]^3$.
  - Edges are line segments; facets are *planar* convex polygons.



What's a "hexahedron"?

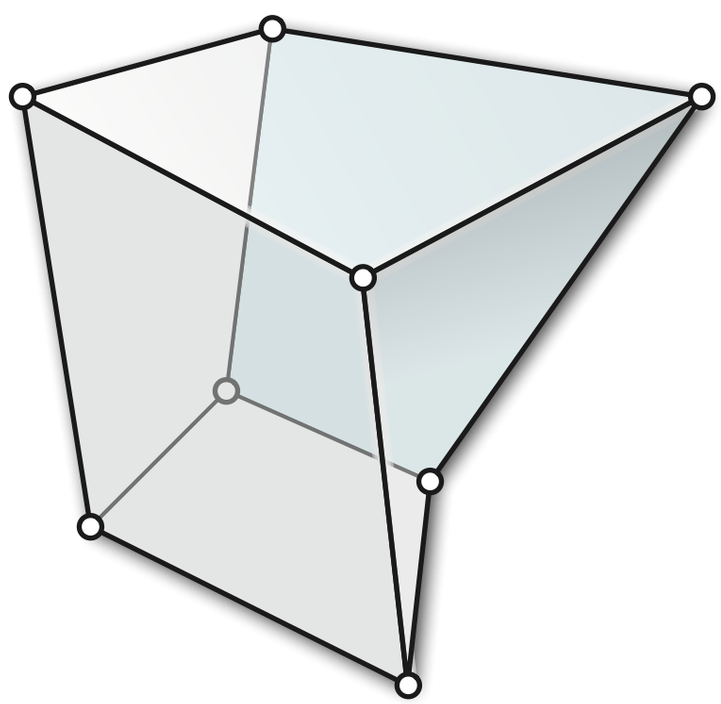
- **Polyhedral:** What computational geometers mean
  - A *hexahedron* is a convex polytope isomorphic to the cube $[0,1]^3$.
  - Edges are line segments; facets are *planar* convex polygons.

Not "approximately planar"
Not "planar up to engineering tolerances"
Not "planar up to floating-point error"



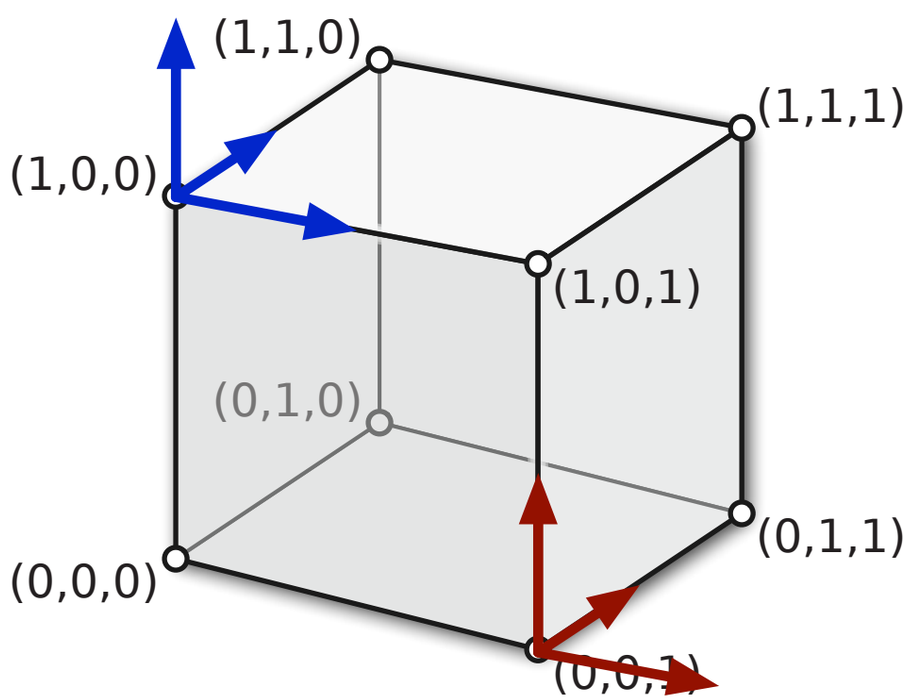
What's a "hexahedron"?

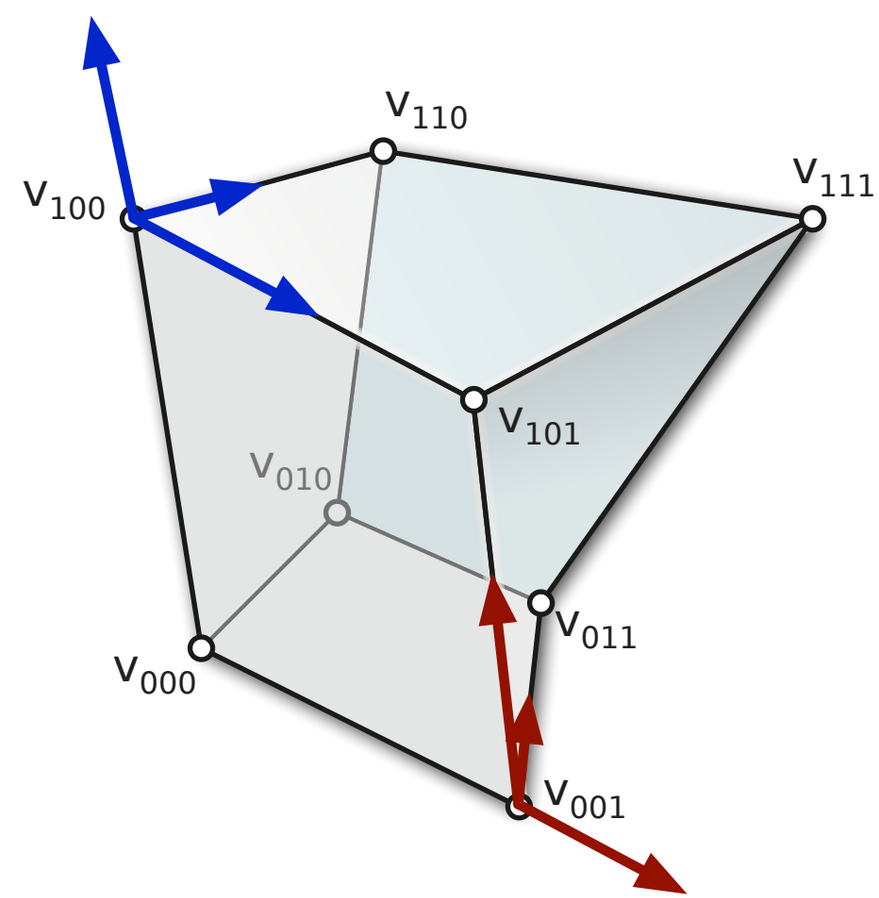
- **Multilinear:** What engineers mean
  - A *hexahedral finite element* is the "multilinear hull" of 8 points in $\boldsymbol{R}^3$.
  - Edges are line segments; facets are *ruled surface patches*



Jacobian $J = \nabla q$

- 3×3 matrix of partial derivatives of multilinear map $q : [0,1]^3 \hookrightarrow \boldsymbol{R}^3$
- Most numerical methods require $\det J > 0$ everywhere
- Equivalently: *locally convex* at every vertex





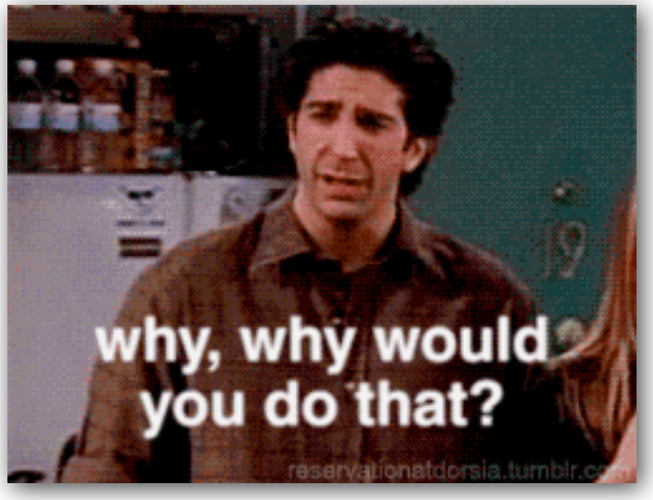
What's a "hex mesh"?

- **Standard cube complex:** A collection of cubes such that
  - Interiors are disjoint.
  - Union is the desired volume.
  - Intersection of any two cubes is a common facet, a common edge, a common vertex, or nothing.

What's a "hex mesh"?

- **Standard cube complex:** A collection of cubes such that
  - Interiors are disjoint.
  - Union is the desired volume.
  - Intersection of any two cubes is a common facet, a common edge, a common vertex, or nothing.

- **WARNING! Some papers use more relaxed definitions!**
  - Two cubes can share multiple faces
  - Shared faces need not be connected
  - A single cube can be adjacent to itself
  - Some hexes can have zero or negative Jacobians

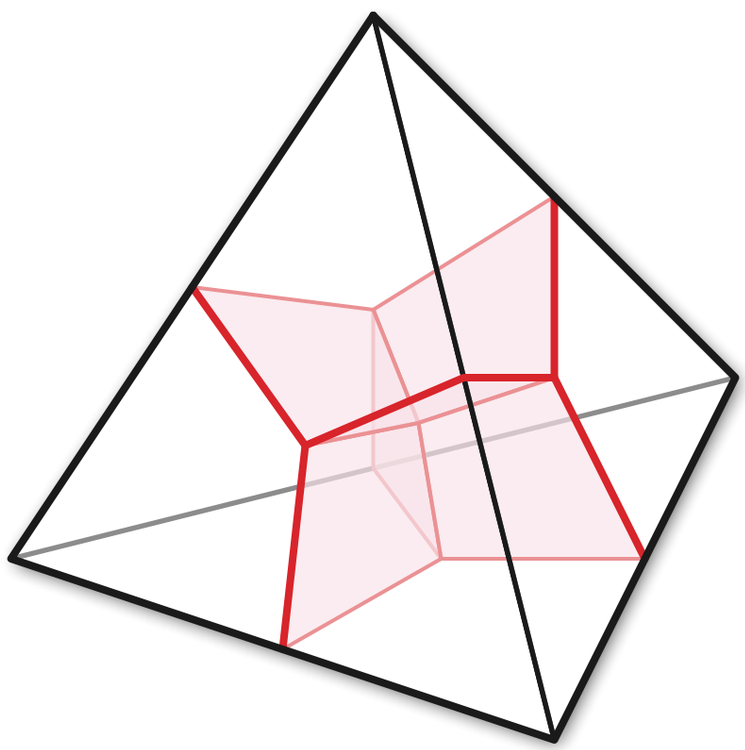


What is the hex meshing problem?

- **First attempt:** Given a 3d volume, subdivide it into a valid hex mesh.

What is the hex meshing problem?

- **First attempt:** Given a 3d volume, subdivide it into a valid hex mesh.
- ***Solved!*** Triangulate, then split each tetrahedron into four *hexahedra*!



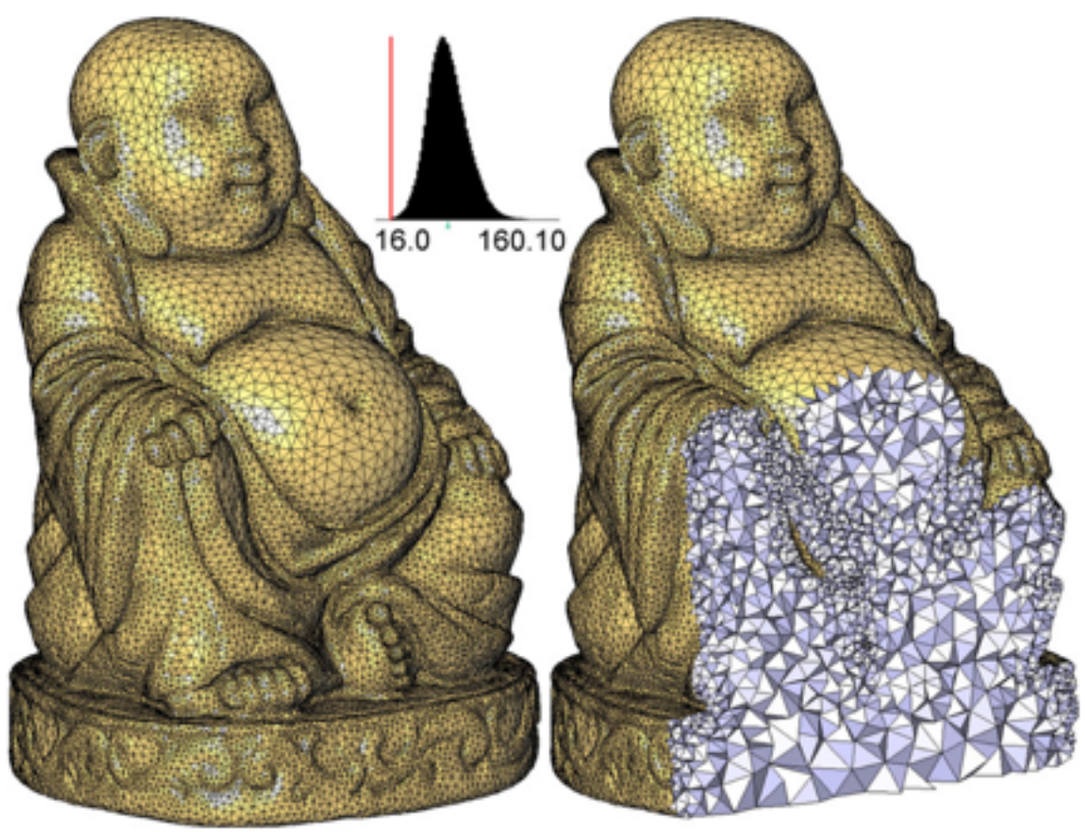
What is the hex meshing problem?

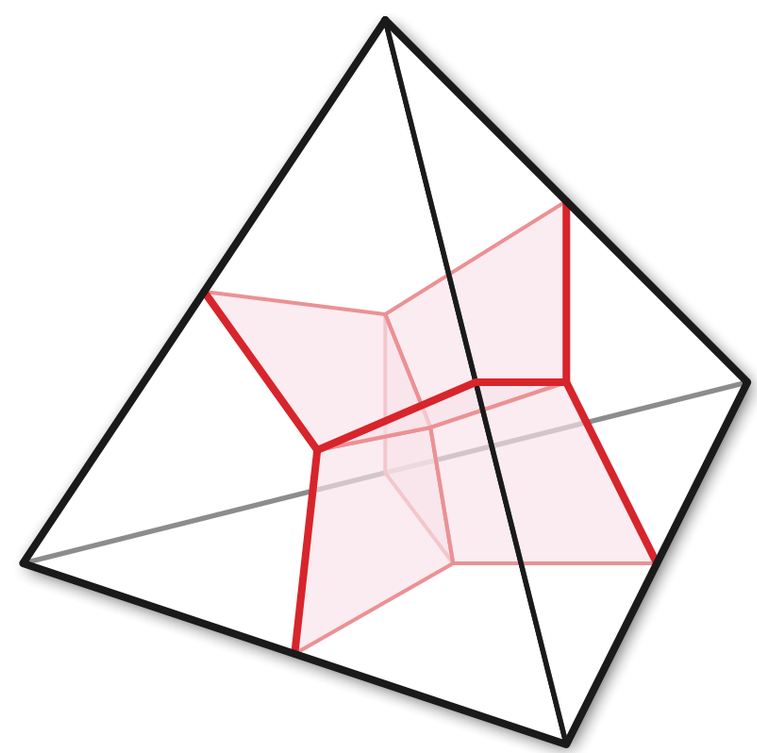
- **Second attempt:** Given a 3d volume, subdivide it into a valid hex mesh, *where every hex has some guaranteed quality*.

What is the hex meshing problem?

- **Second attempt:** Given a 3d volume, subdivide it into a valid hex mesh, *where every hex has some guaranteed quality*.

- ***Solved!*** Compute a tet mesh with guaranteed quality, then split each tetrahedron into four hexahedra!





What is the hex meshing problem?

- **Third attempt:** Given a 3d volume, subdivide it into a ***useful*** hex mesh, where every hex has at least some guaranteed ***practical*** quality.

What is the hex meshing problem?

- **Third attempt:** Given a 3d volume, subdivide it into a ***useful*** hex mesh, where every hex has at least some guaranteed ***practical*** quality.

- ***This is open!***

What is the hex meshing problem?

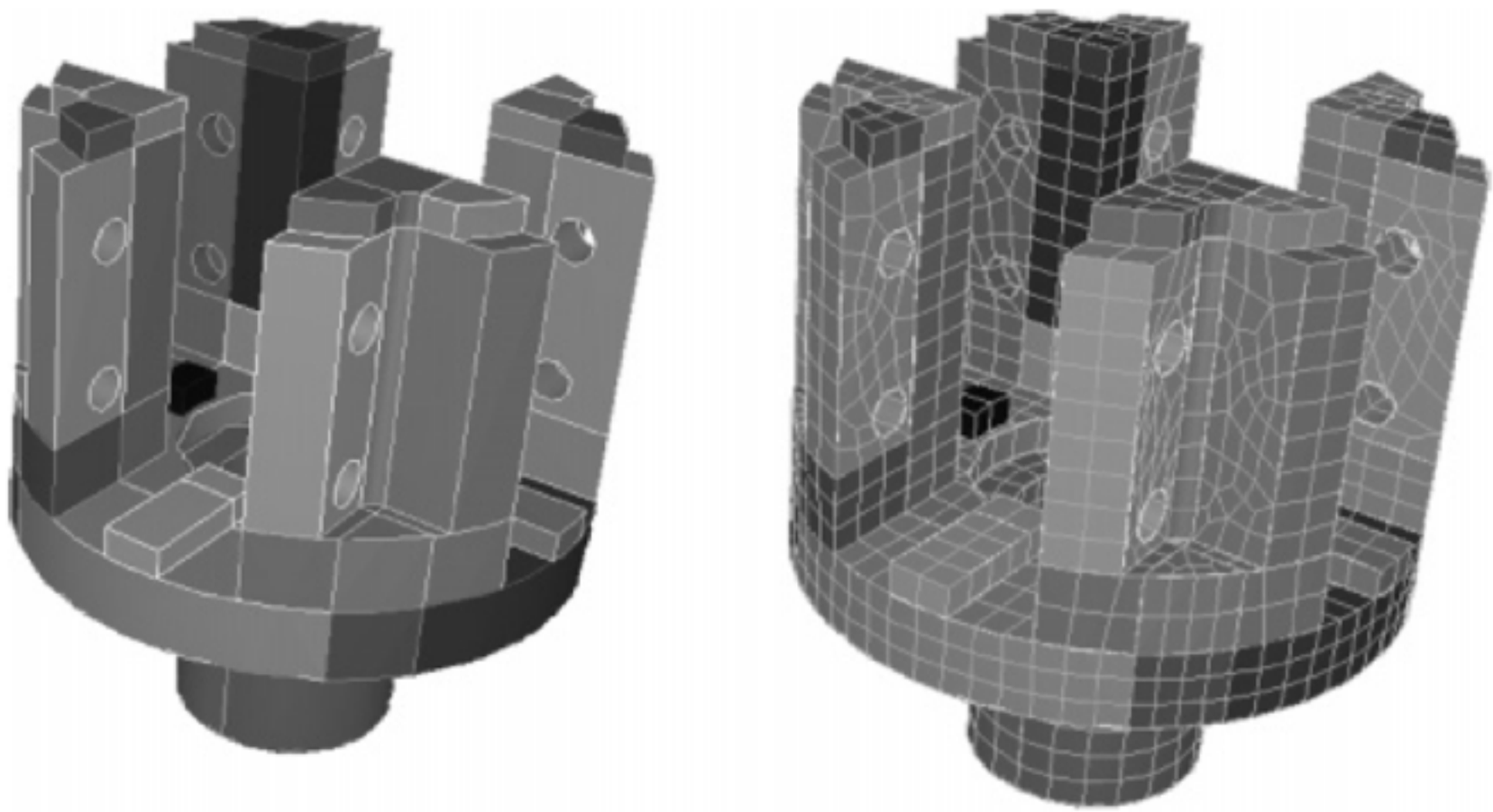
- **Third attempt:** Given a 3d volume, subdivide it into a ***useful*** hex mesh, where every hex has at least some guaranteed ***practical*** quality.

- ***This is open!***

- But what do "useful" and "practical quality" actually mean?
  - Without crisp, unambiguous definitions, the problem is ill-defined.
  - Computational geometers like meshes with flat faces, but "skew" (variance from flatness) is only one of many possible quality measures, and probably not the most important one.

Practical techniques

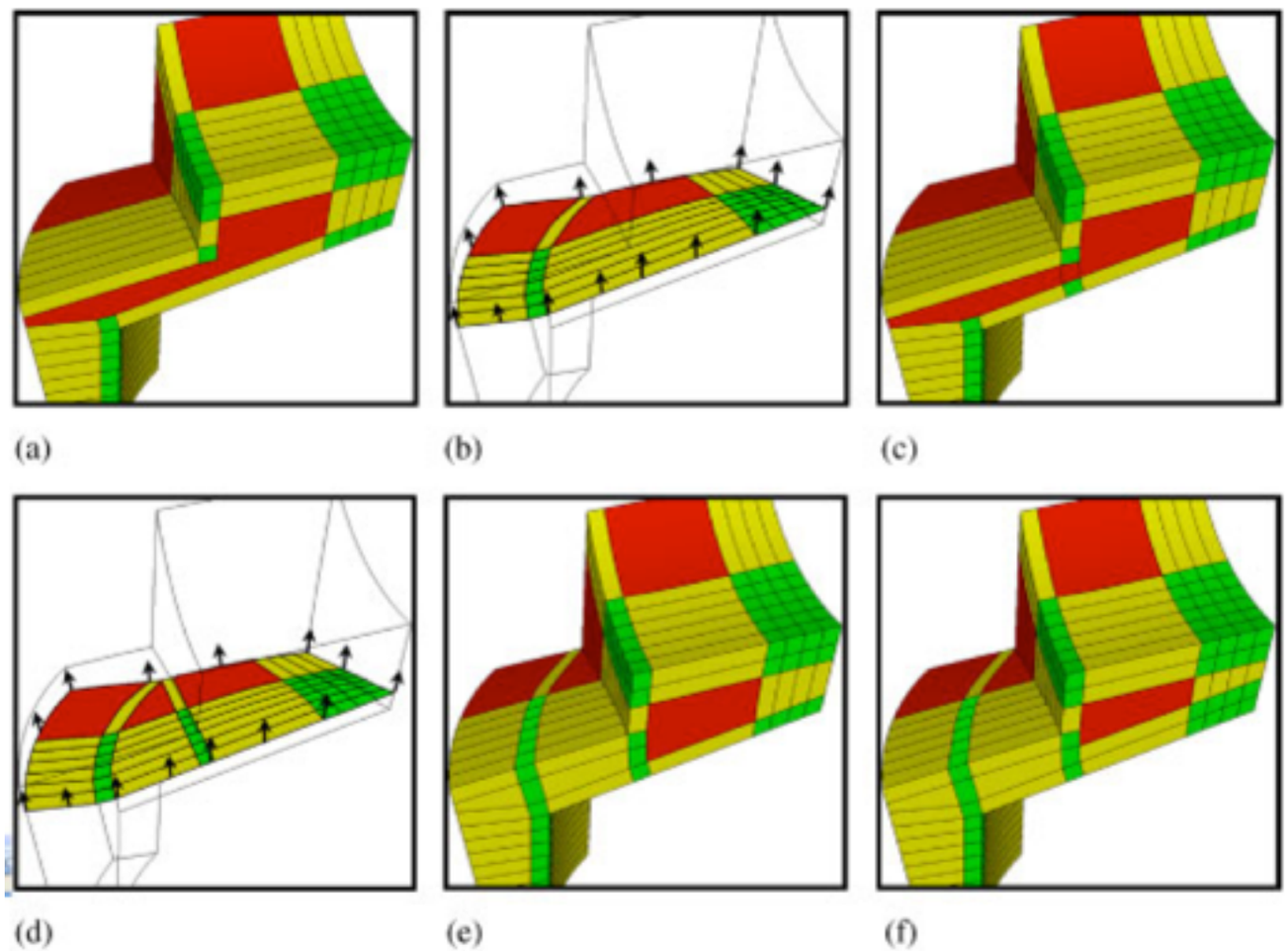
- Decomposition: Break the model into pieces that are easier to mesh



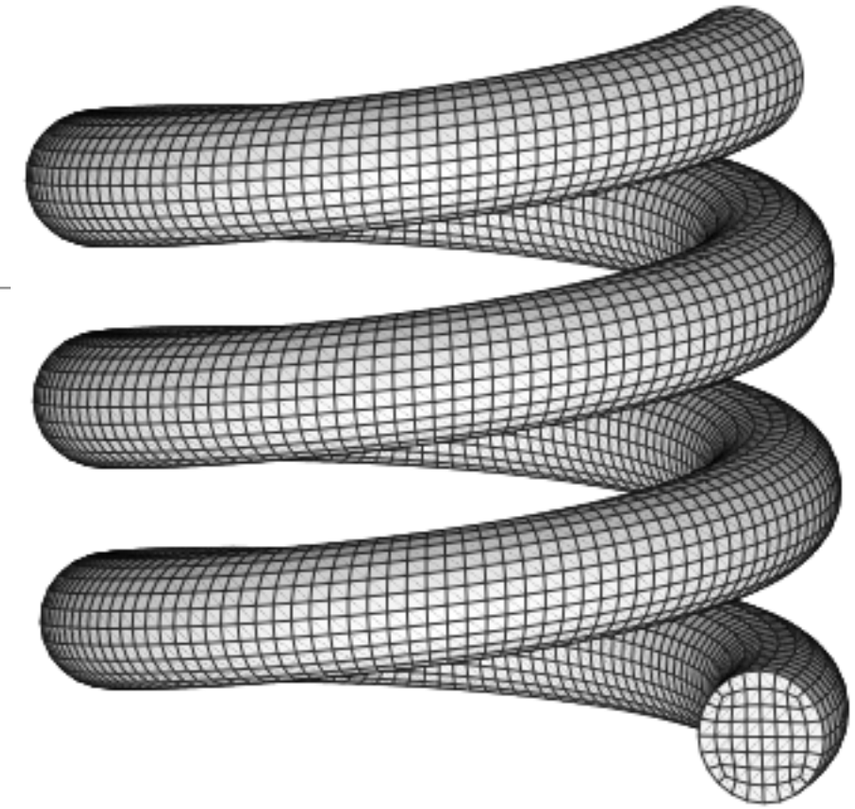
[Tautges '01]

Practical techniques

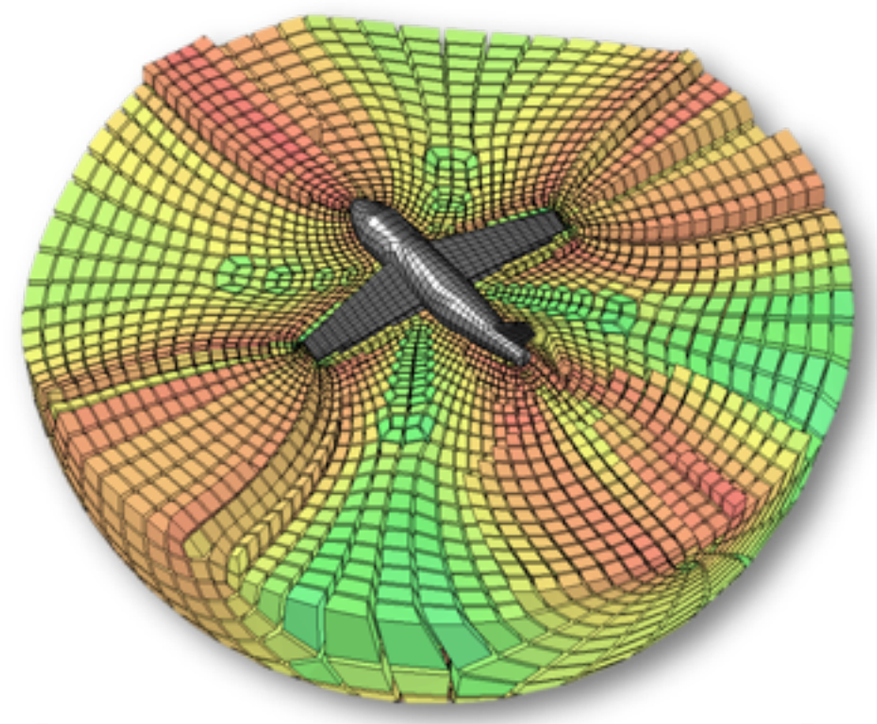
- Advancing fronts: sweeping, paving, plastering, etc.



[Staten Kerr Owen Blacker Stupazzini Shimada '11]



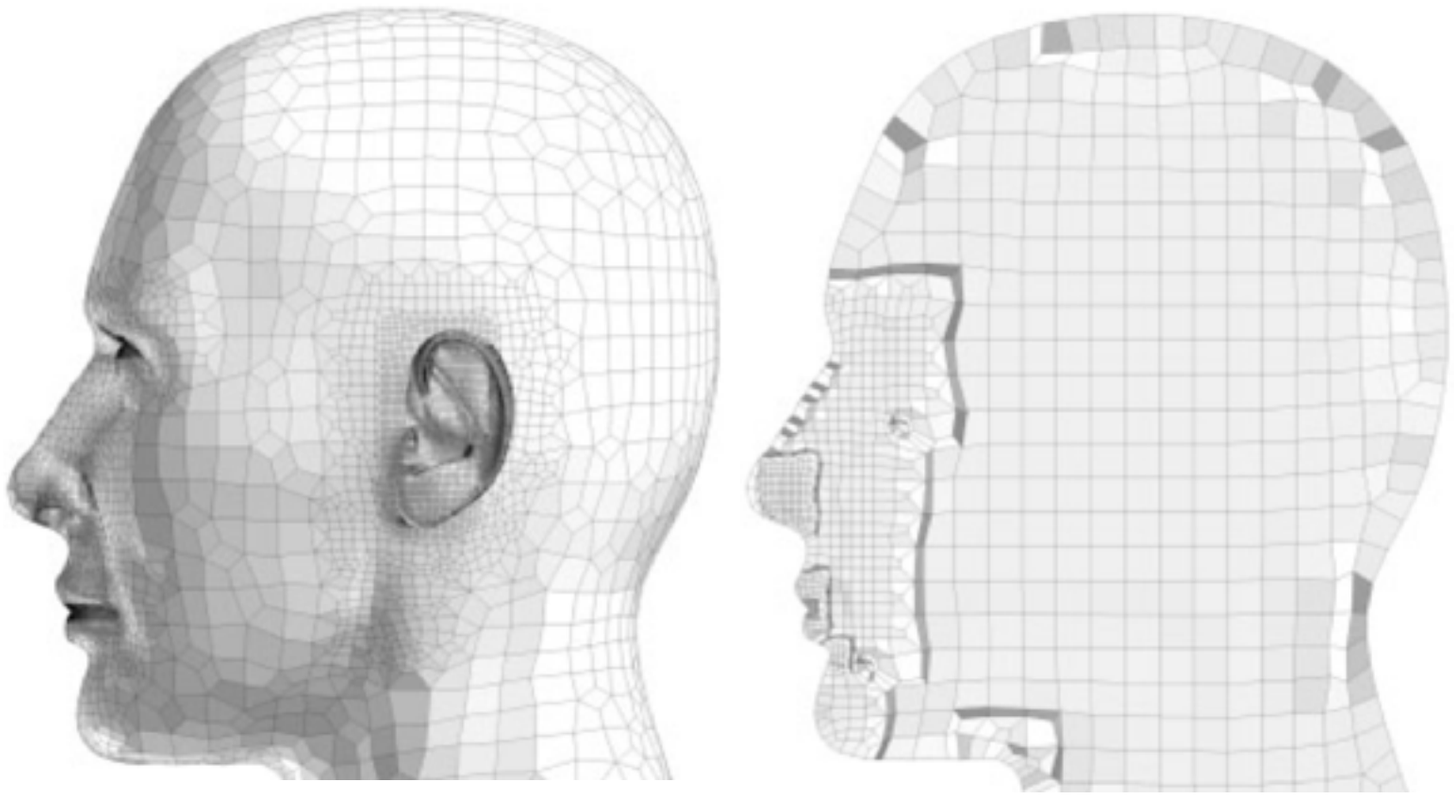
[Roca Sarrate Huerte 2004]



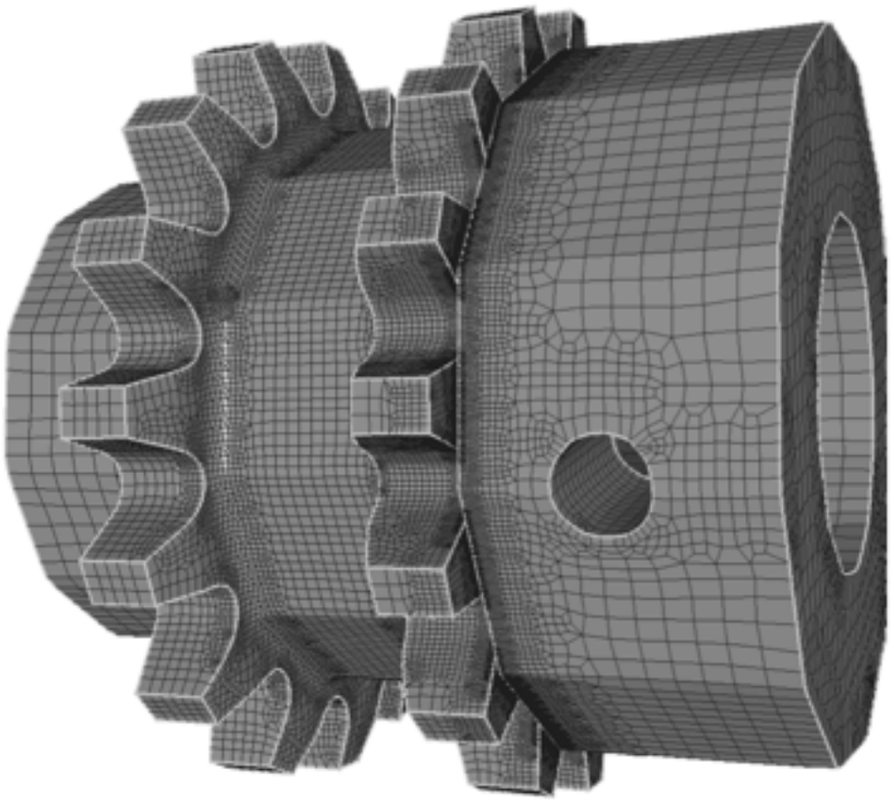
[Ruiz-Gironès Roca Sarrate 2012]

Practical techniques

- Grids and octtrees



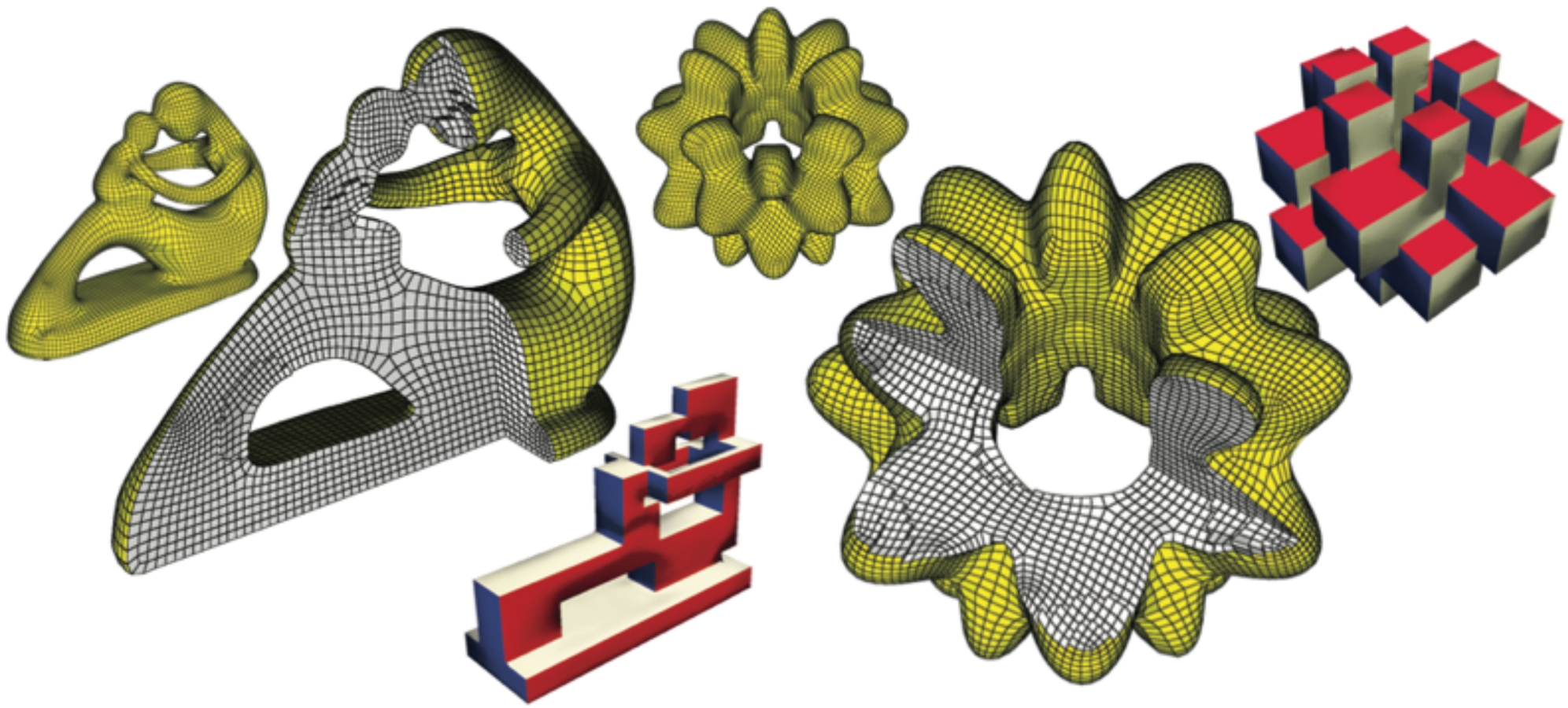
[Ito Shih Soni '08]



[Maréchal '09]

Practical techniques

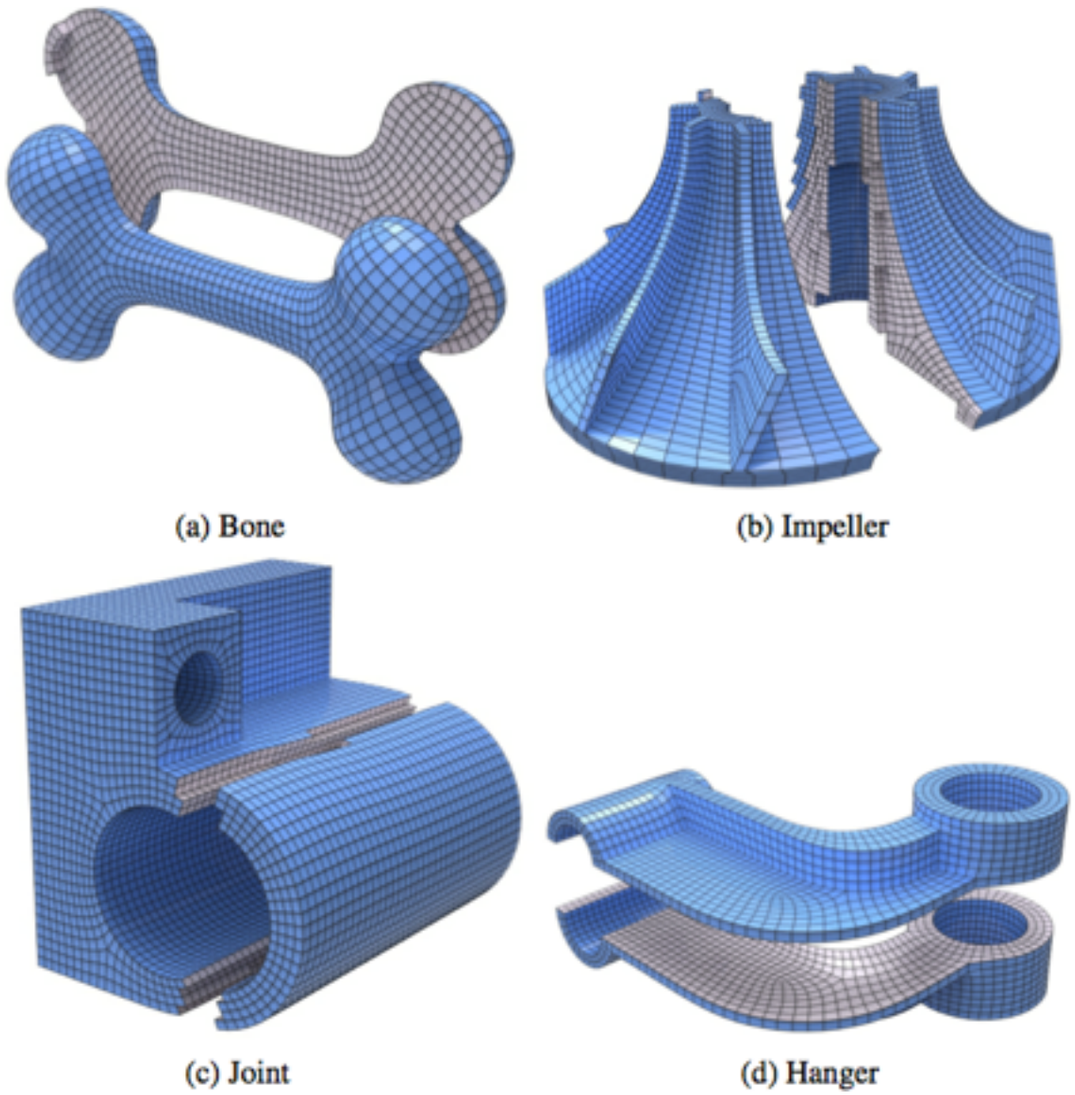
- Mapping



[Gregson Sheffer Zhang '11]

Practical techniques

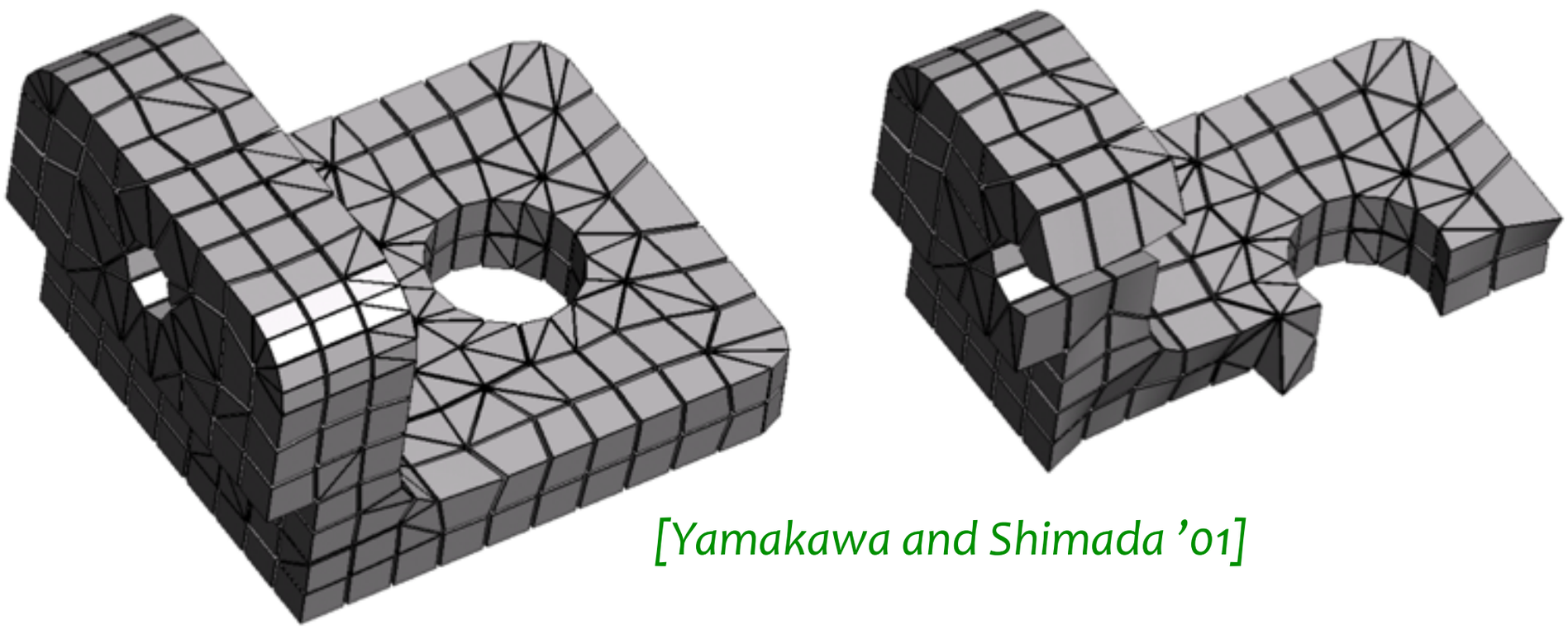
- 3D parametrization / frame fields



[Li Liu Xu Wang Guo '12]

Subdivision methods

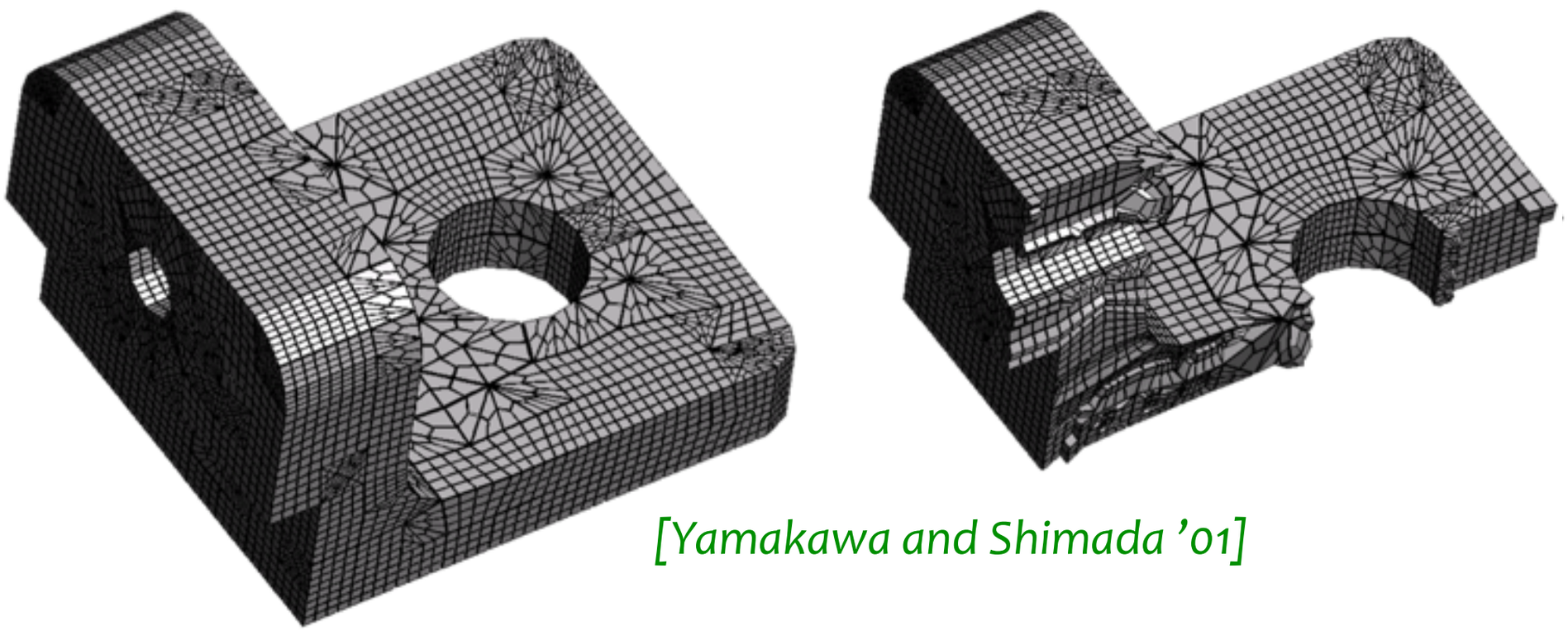
- Default to tetrahedra when direct hex-meshing methods break down. Possibly include transitional pyramid and prism elements.
- Then refine mixed mesh into a hex mesh.



[Yamakawa and Shimada '01]

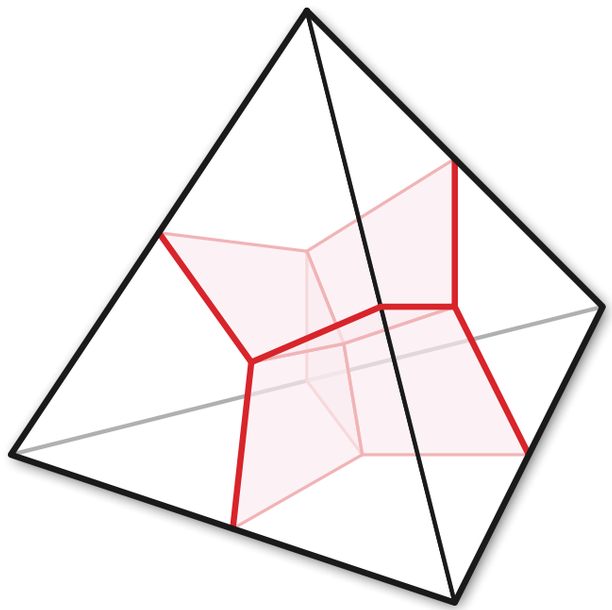
Subdivision methods

- Default to tetrahedra when direct hex-meshing methods break down. Possibly include transitional pyramid and prism elements.
- Then refine mixed mesh into a hex mesh.

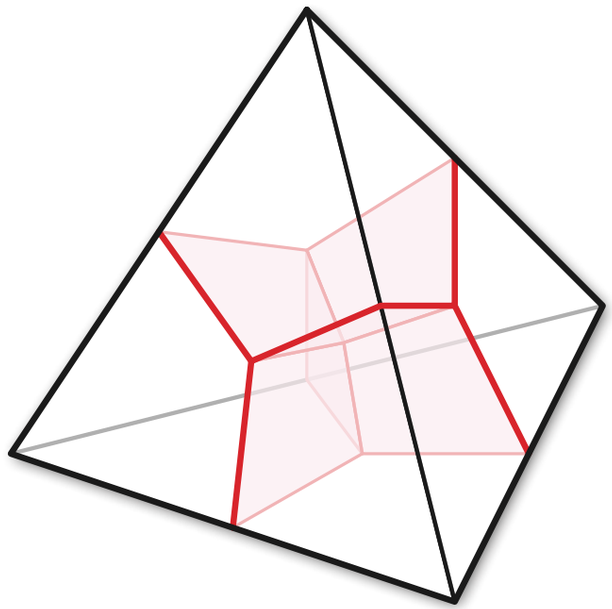


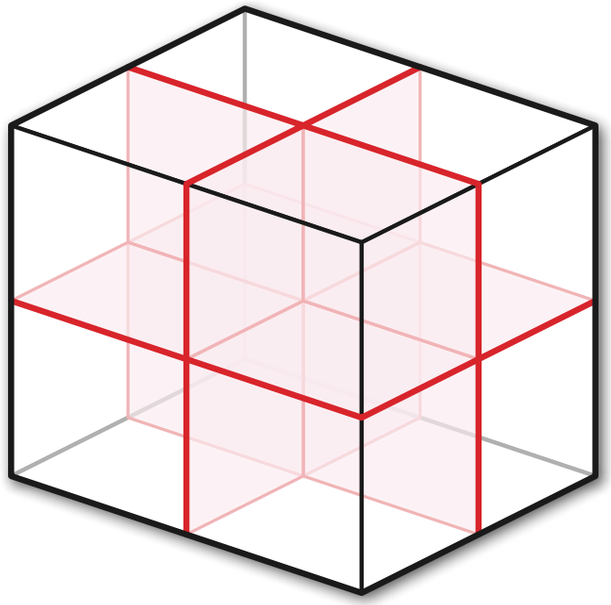
[Yamakawa and Shimada '01]

First try at refinement

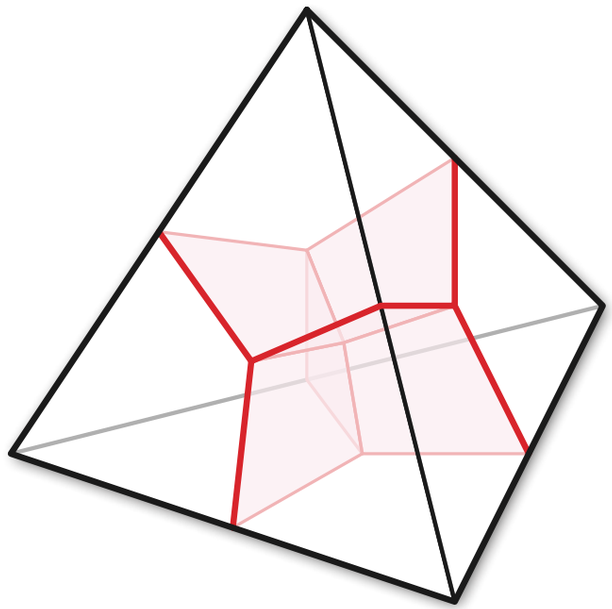


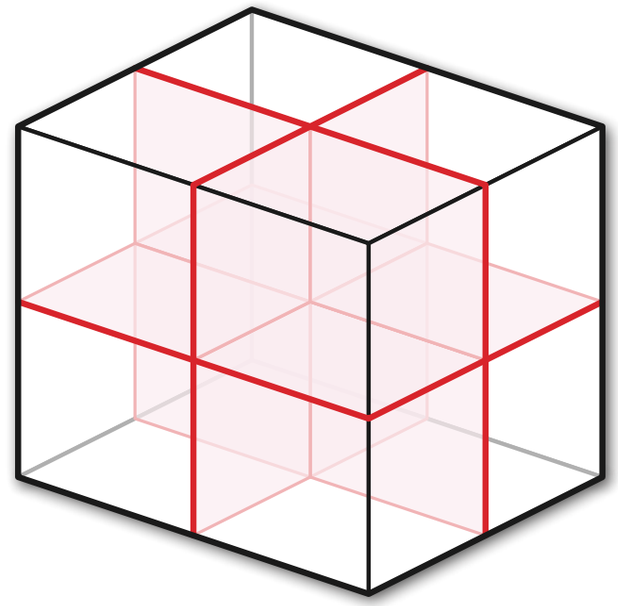
First try at refinement

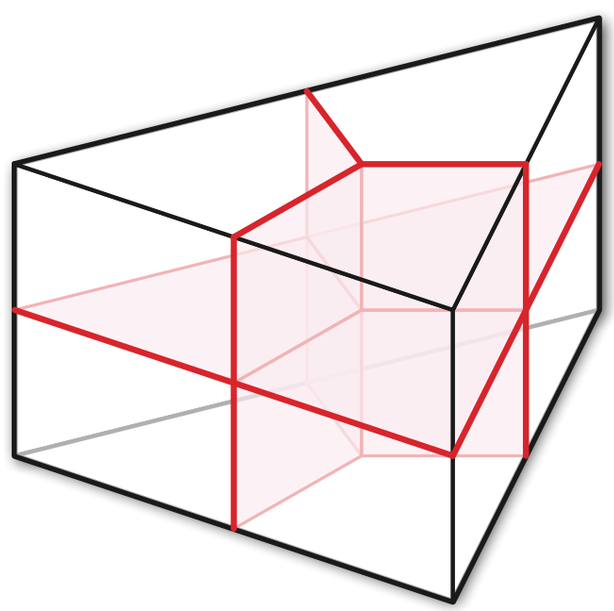




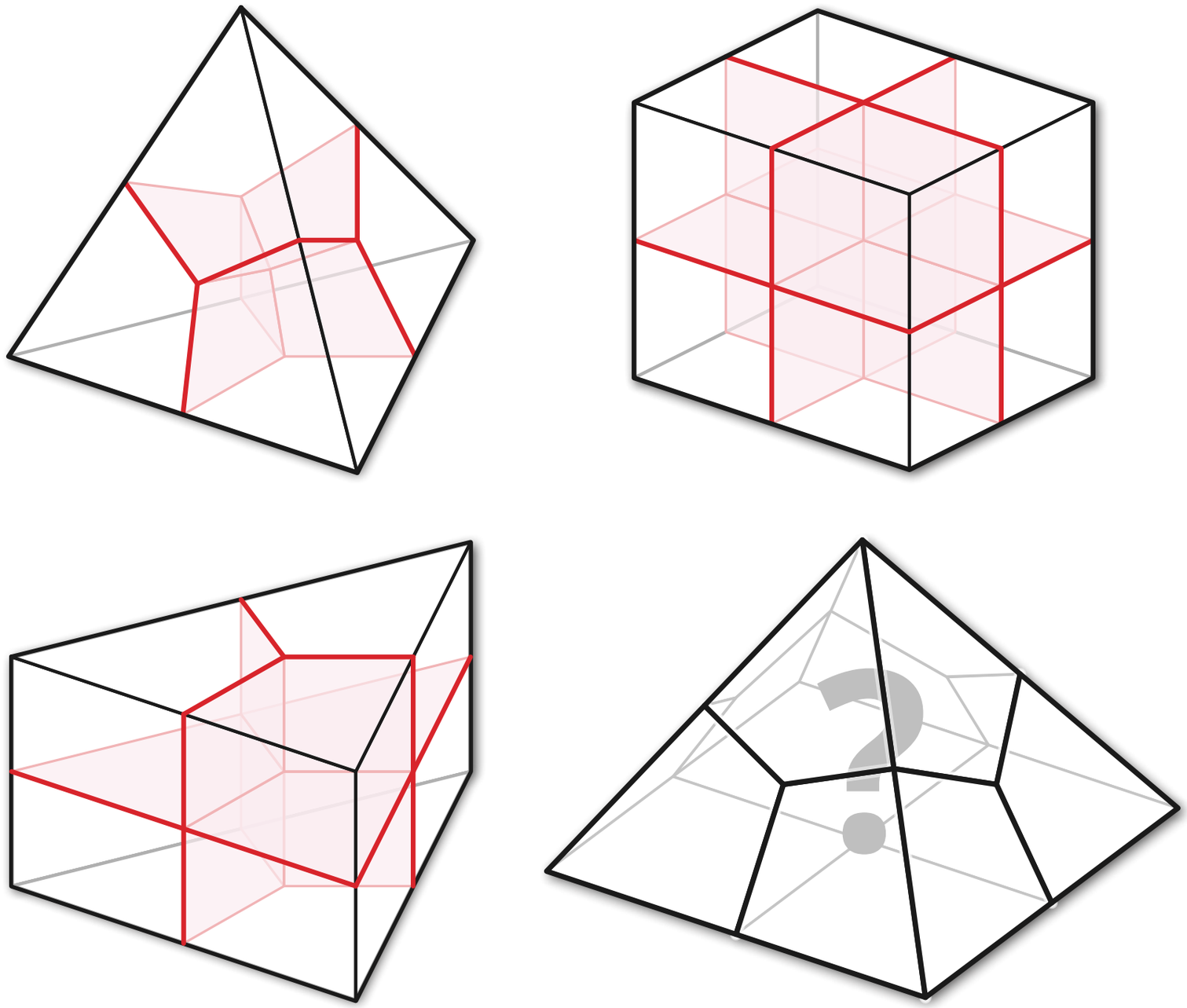
First try at refinement







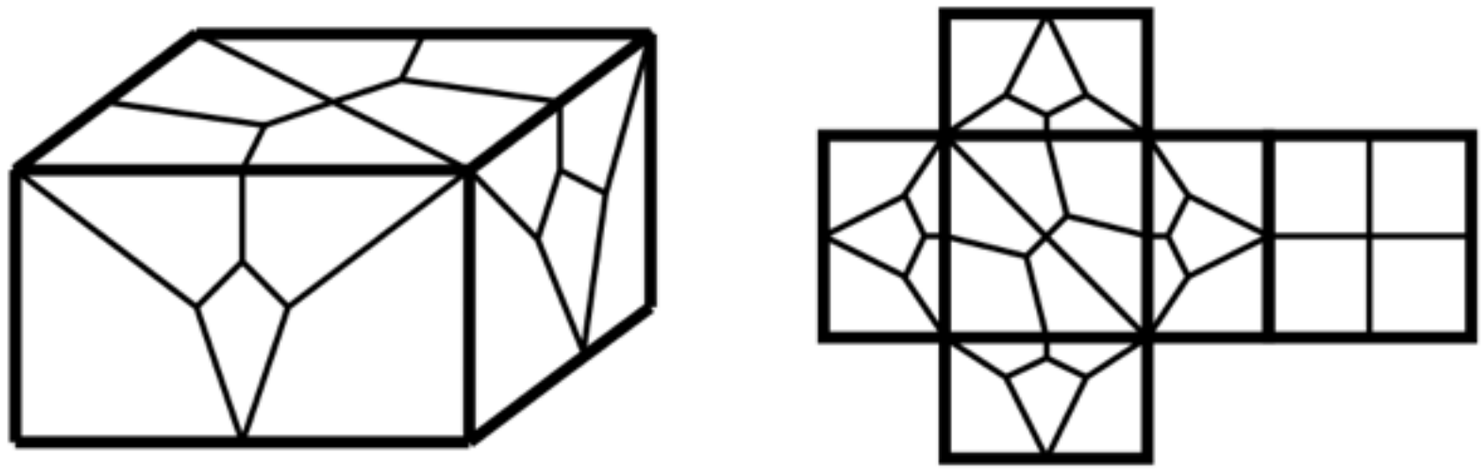
First try at refinement



[Schneiders '95]

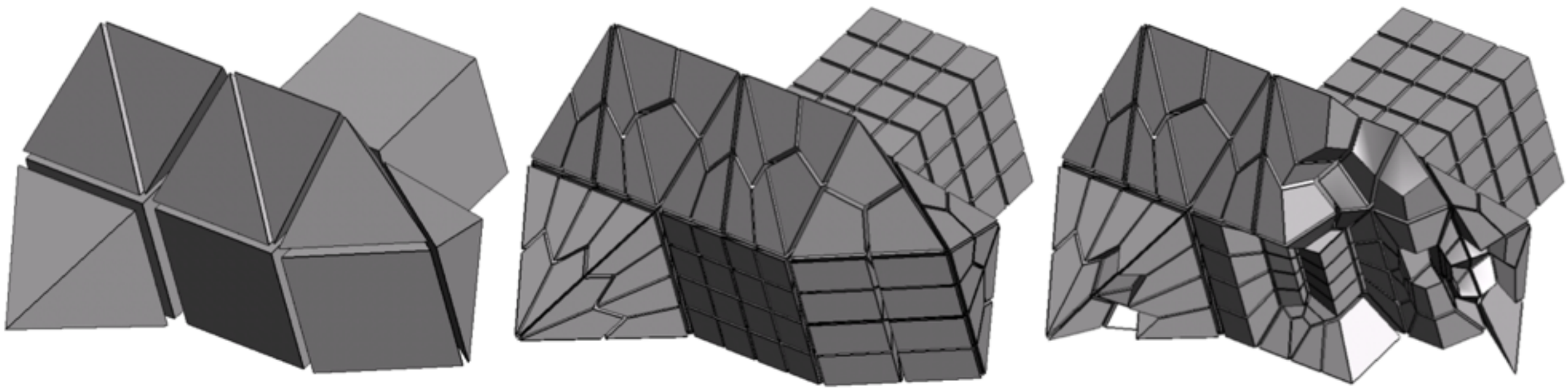
Mitchell's Geode

- Transition layer between tets and hexes. *[Mitchell '98]*



Yamakawa and Shimada's HexHoop

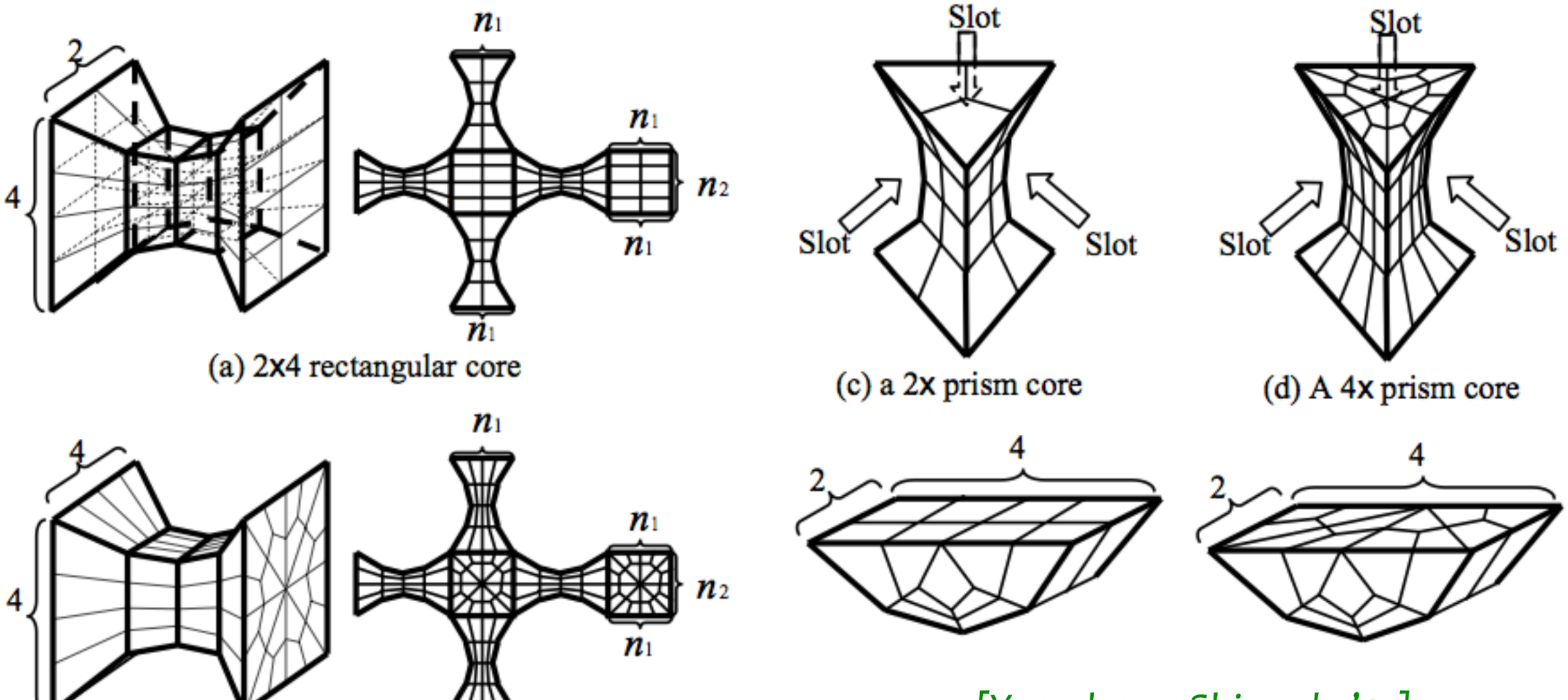
- General templates for converting mixed meshes to hex meshes



[Yamakawa Shimada '01]

HexHoop

- Split pyramid into tets; use standard template for tets
- Templates for cubes and prisms depend on neighboring cells.



(a) 2x4 rectangular core

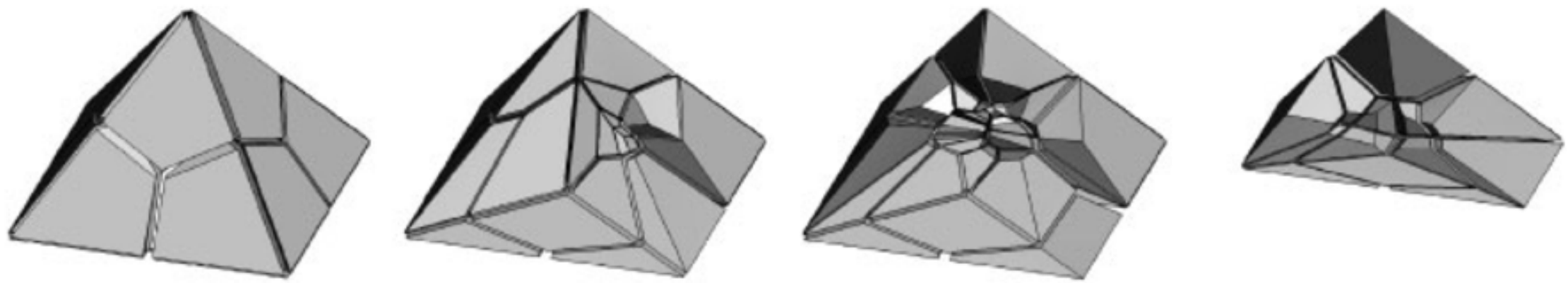
(b) 4x4 triangular core

(c) a 2x prism core

(d) A 4x prism core

[Yamakawa Shimada '01]
US Patent 6,625,938 (Sept. 30, 2003)

88-hex Schneider's pyramid

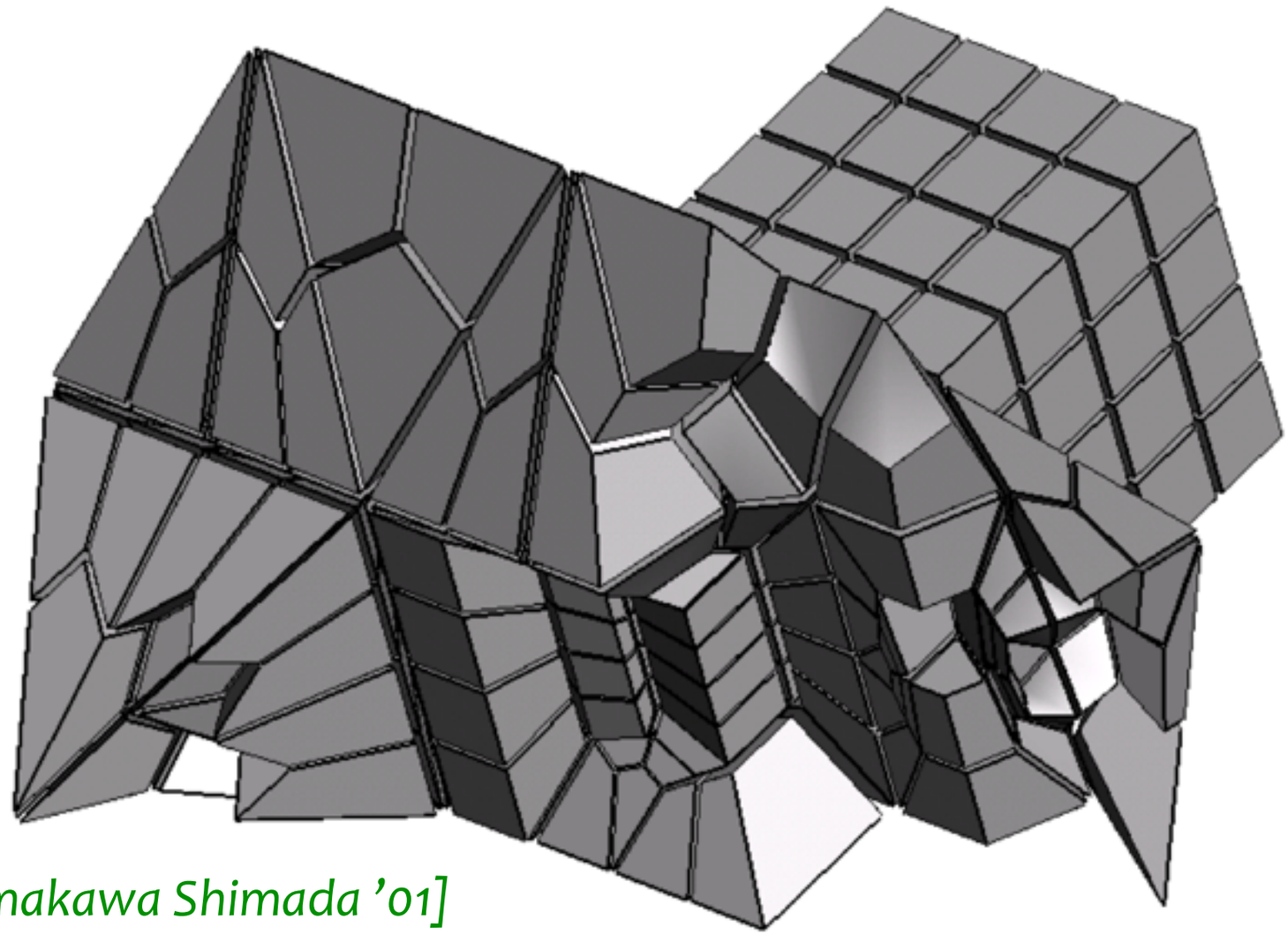


[Yamakawa Shimada '10]

- Standard cube complex: Hexes meet properly face to face. ✓
- Positive Jacobians: No twisted, inverted, or degenerate hexes. ✓
- Internal faces are (slightly) warped.

HexHoop

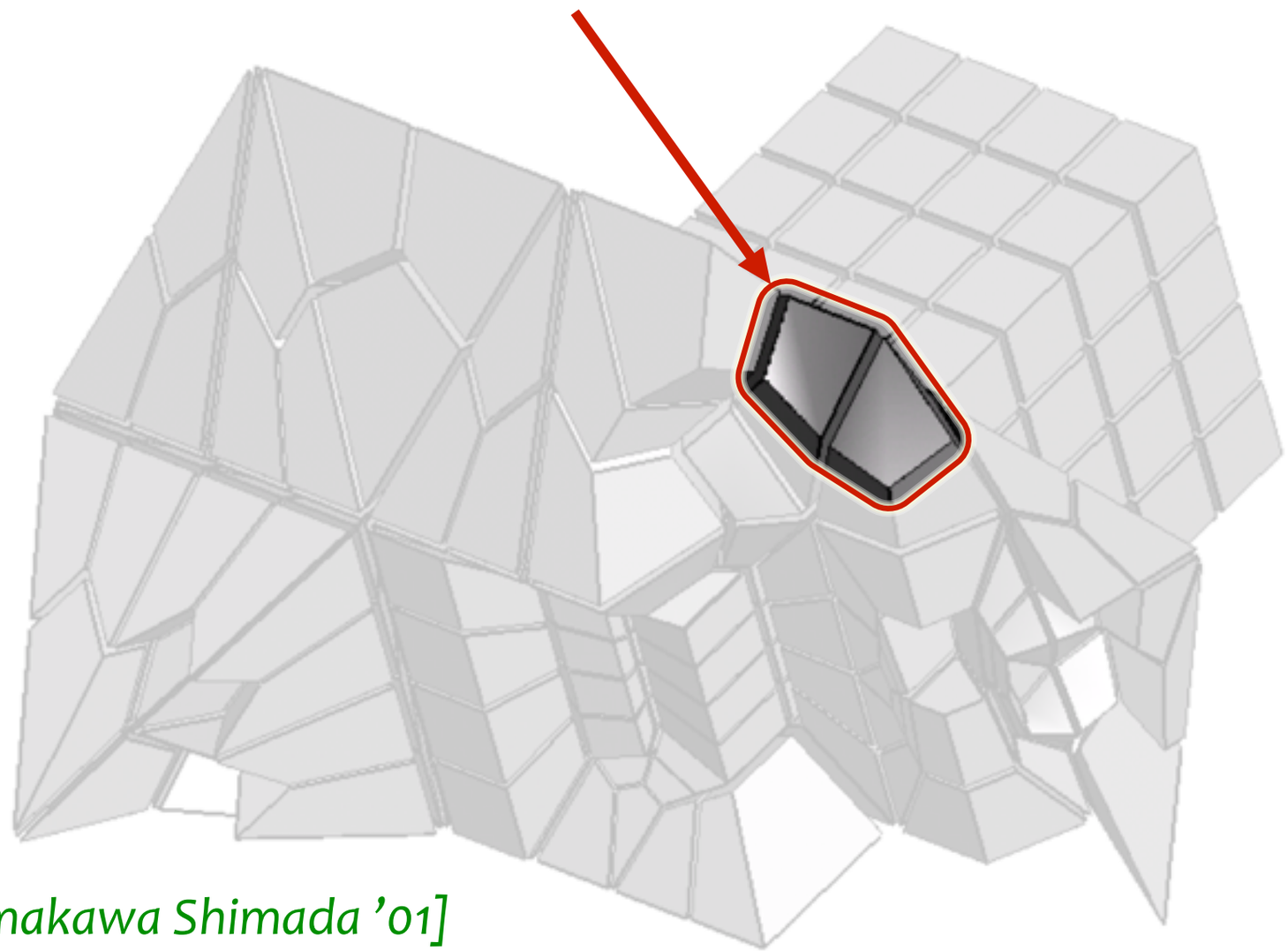
- Internal faces are (slightly) warped.



[Yamakawa Shimada '01]

HexHoop

- Internal faces are (slightly) warped.



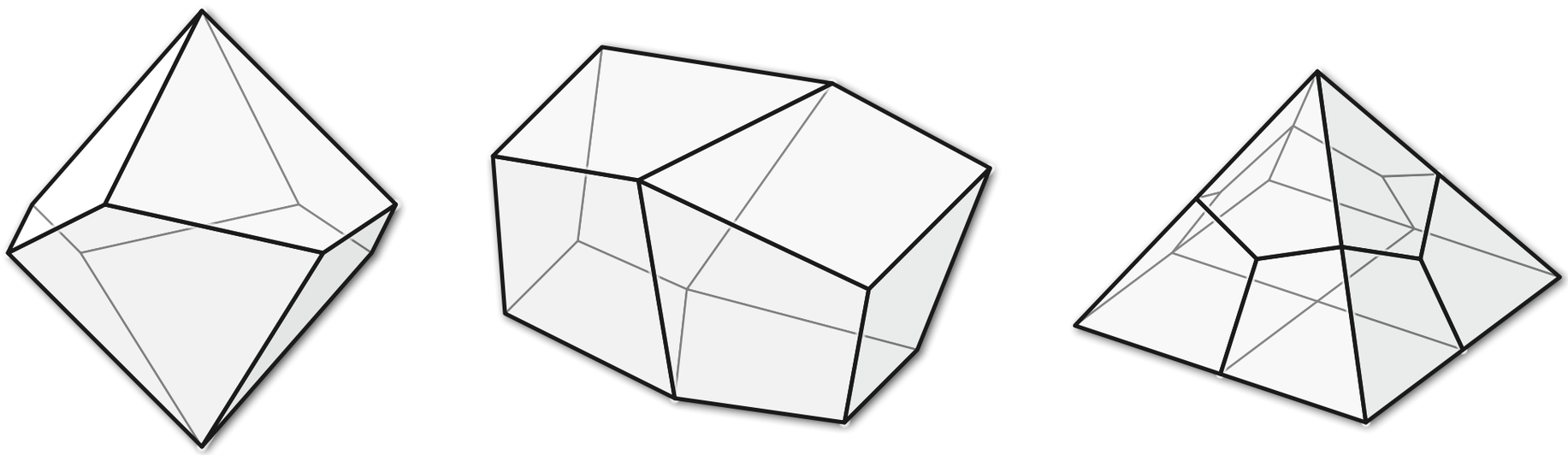
[Yamakawa Shimada '01]

No boundary refinement

- **New problem:** Subdivide a given volume into a valid hex mesh *whose boundary is equal to a given boundary quad mesh*.
  - After decomposition, need to mesh each component independently
    - Parts agree on common boundary meshes in advance
  - Some applications require greater accuracy at the boundary.
    - Separately generate a high-quality boundary mesh, then don't change it.

No boundary refinement

- **Open** for polyhedral meshes, even for the simplest nontrivial inputs.



Octagonal spindle Bicuboid Schneiders' pyramid

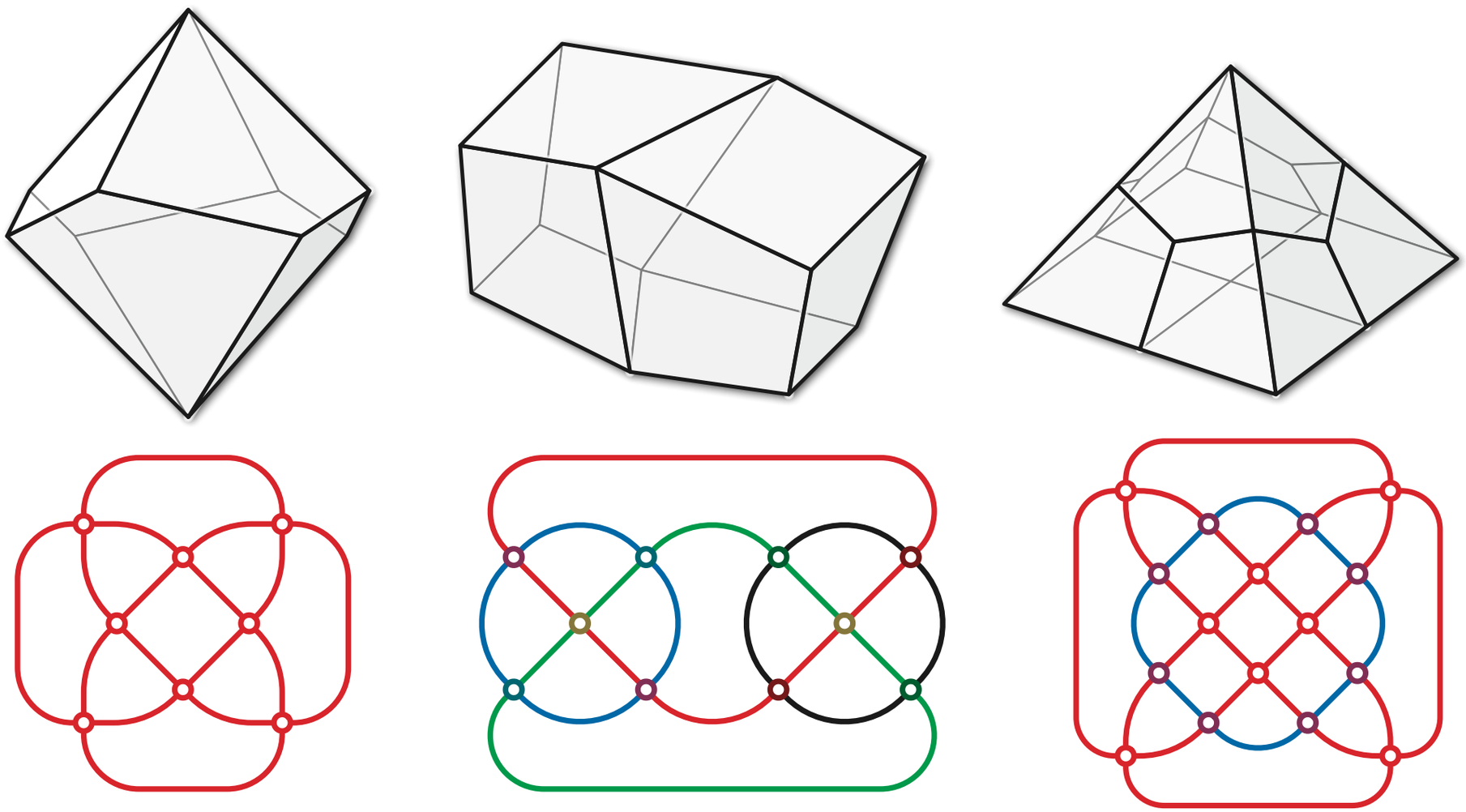
- **Closed** for topological meshes! *[Thurston '93, Mitchell '96, Erickson '13]*

Necessary condition

- **Lemma:** Every hex mesh has an even number of boundary quads.
- **Proof:**
  - Every hex has six boundary quads. Six is even.
  - Gluing two hexes removes two boundary quads. Two is even.

Dual curves

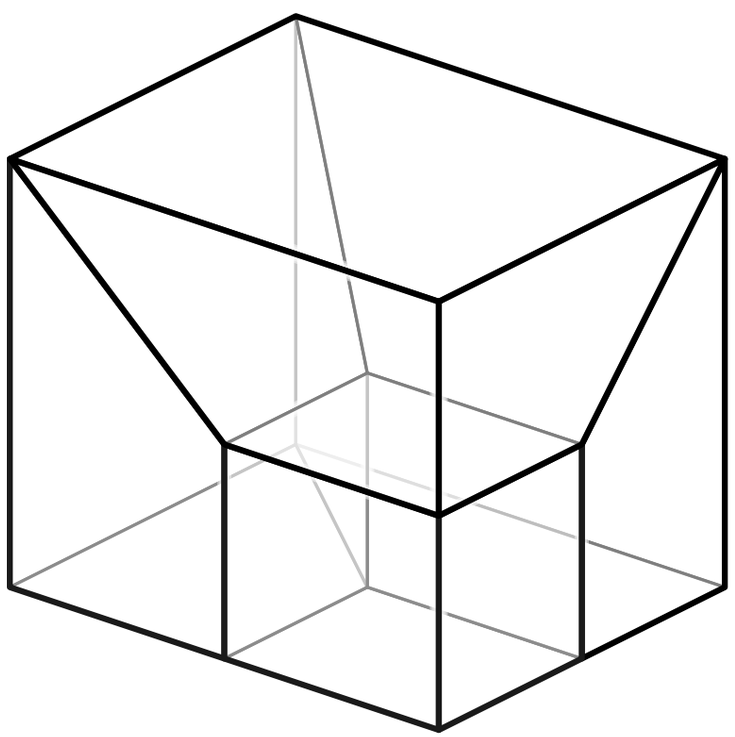
- The dual Q^* of any surface quad mesh Q is an immersion of circles.

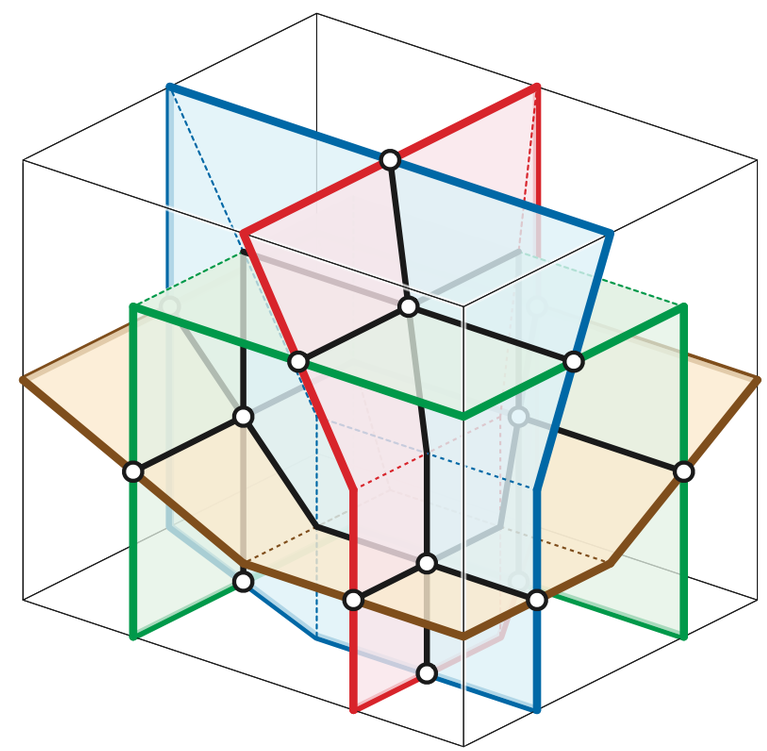


Octagonal spindle Bicuboid Schneiders' pyramid

Dual surfaces

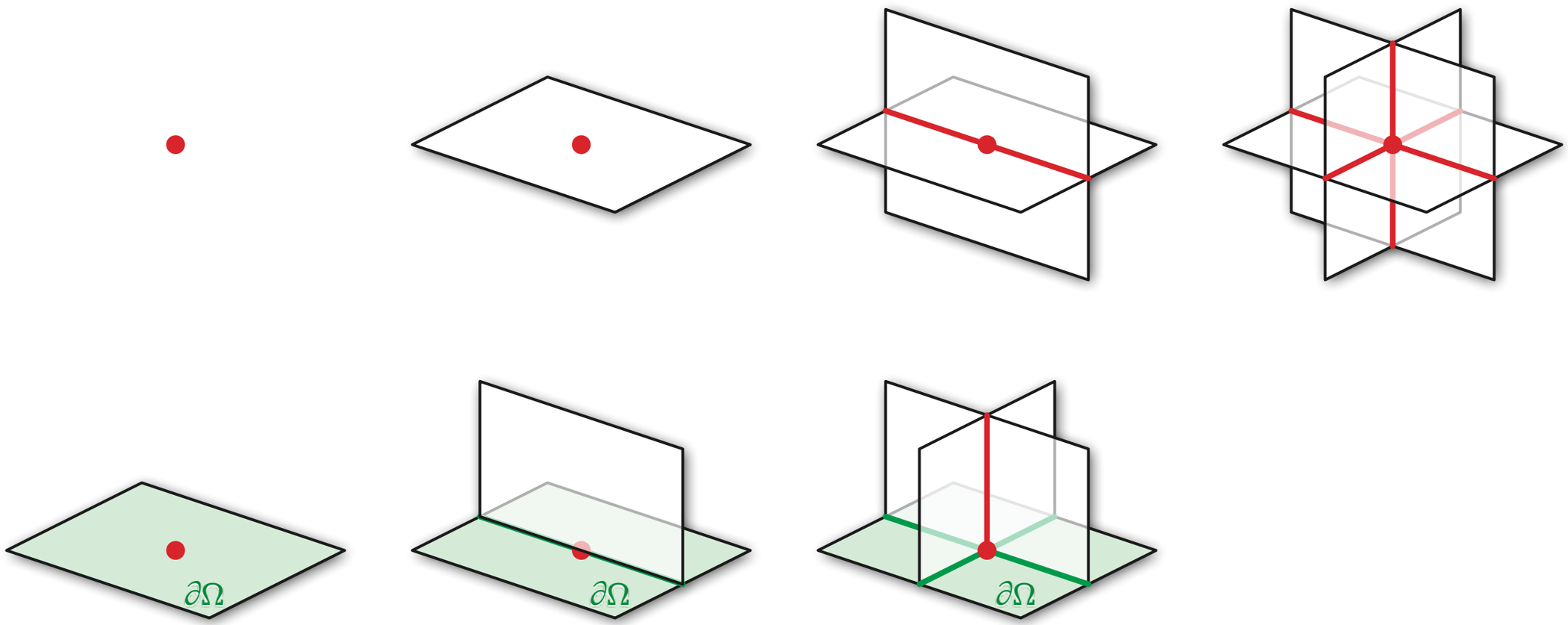
- The dual of any hex mesh is an immersion of surfaces.
  - dual arrangements of zonotopes *[Fedorov 1885]*
  - "derivative complex" *[Jockusch '93 (MacPherson, Stanley)]*
  - "hyperplanes" *[Sageev '95]*
  - "spatial twist continuum" *[Murdoch Benzley '95, Mitchell '96]*
  - "canonical surface" *[Aitchison et al. '97]*





Surface immersion

- Every point in the volume has one of these neighborhoods:



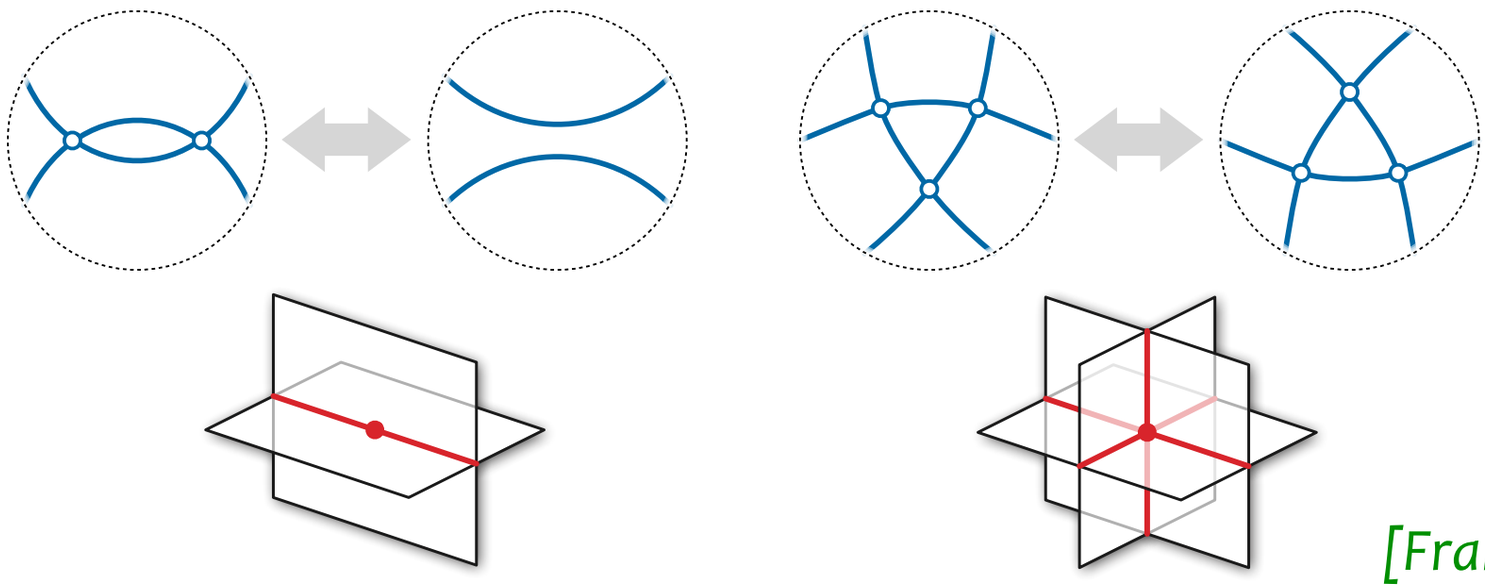
Genus-zero meshes

- **Theorem:** A quad mesh of *the sphere* can be extended to a hex mesh of *the ball* if and only if the number of quads is even.
  [Thurston '93, Mitchell '96]

- **Proof:**
  - Extend dual curves on the sphere to surface immersion in the ball.
    [Csikós Szűcs '95]
  - Refine surface immersion into the dual of hex mesh.

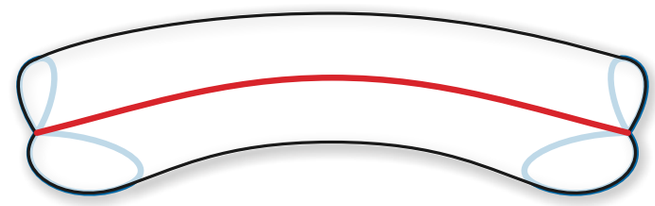
Extending curves to surfaces

- Shrink the sphere inward; the dual curves sweep out surfaces.
- Simplify the curves with the following moves — *regular homotopy*.



[Francis '71, Titus '73]

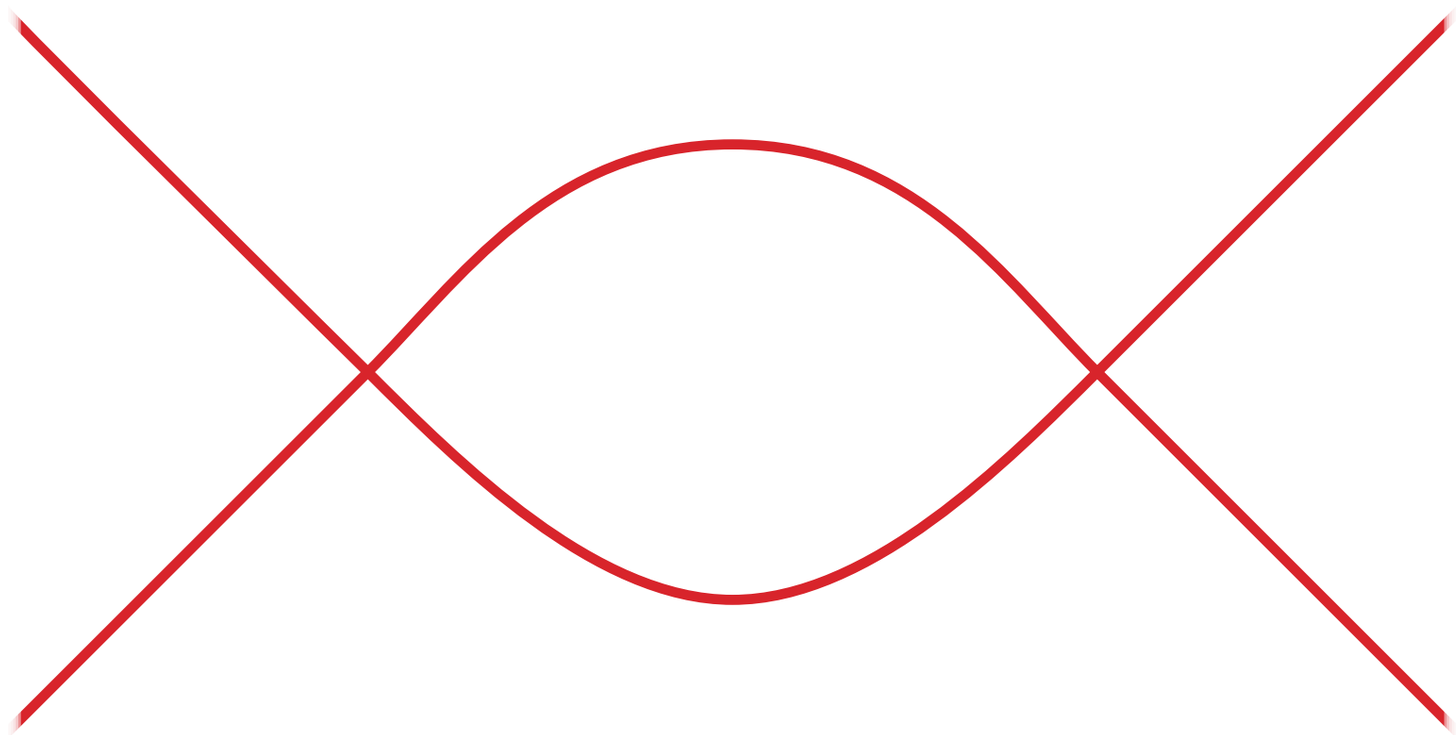
- Simplification yields disjoint circles and figure 8s. *[Boy 1901, Whitney '37]*
- Cap off circles, pair up figure 8s.



[Hass Hughes '92]

Bubble-wrapping

- Any surface immersion can be refined into the dual of a hex mesh.

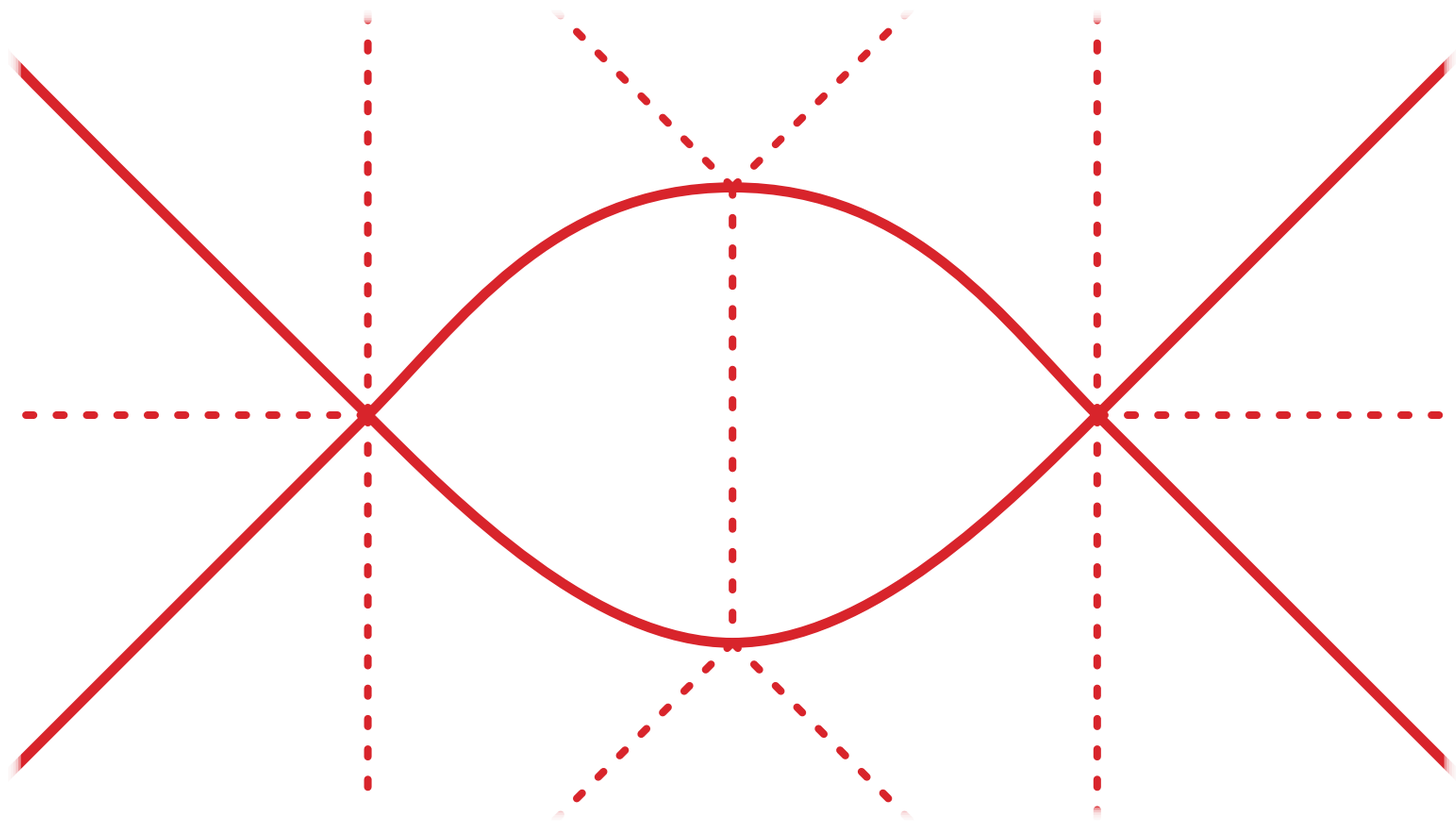


[Babson Chan '08]

- Mitchell gets this slightly wrong.

Bubble-wrapping

- Any surface immersion can be refined into the dual of a hex mesh.

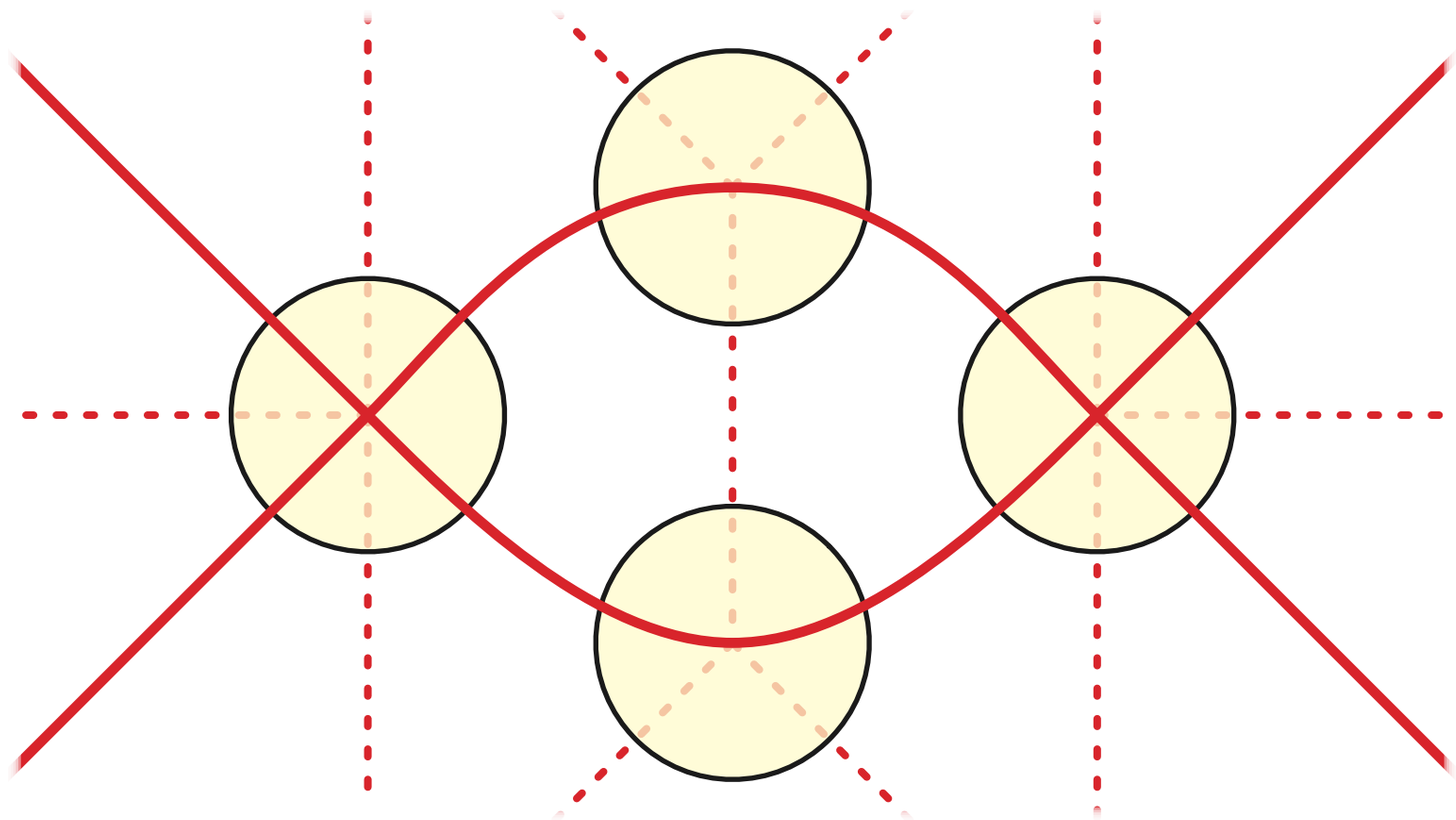


[Babson Chan '08]

- Mitchell gets this slightly wrong.

Bubble-wrapping

- Any surface immersion can be refined into the dual of a hex mesh.

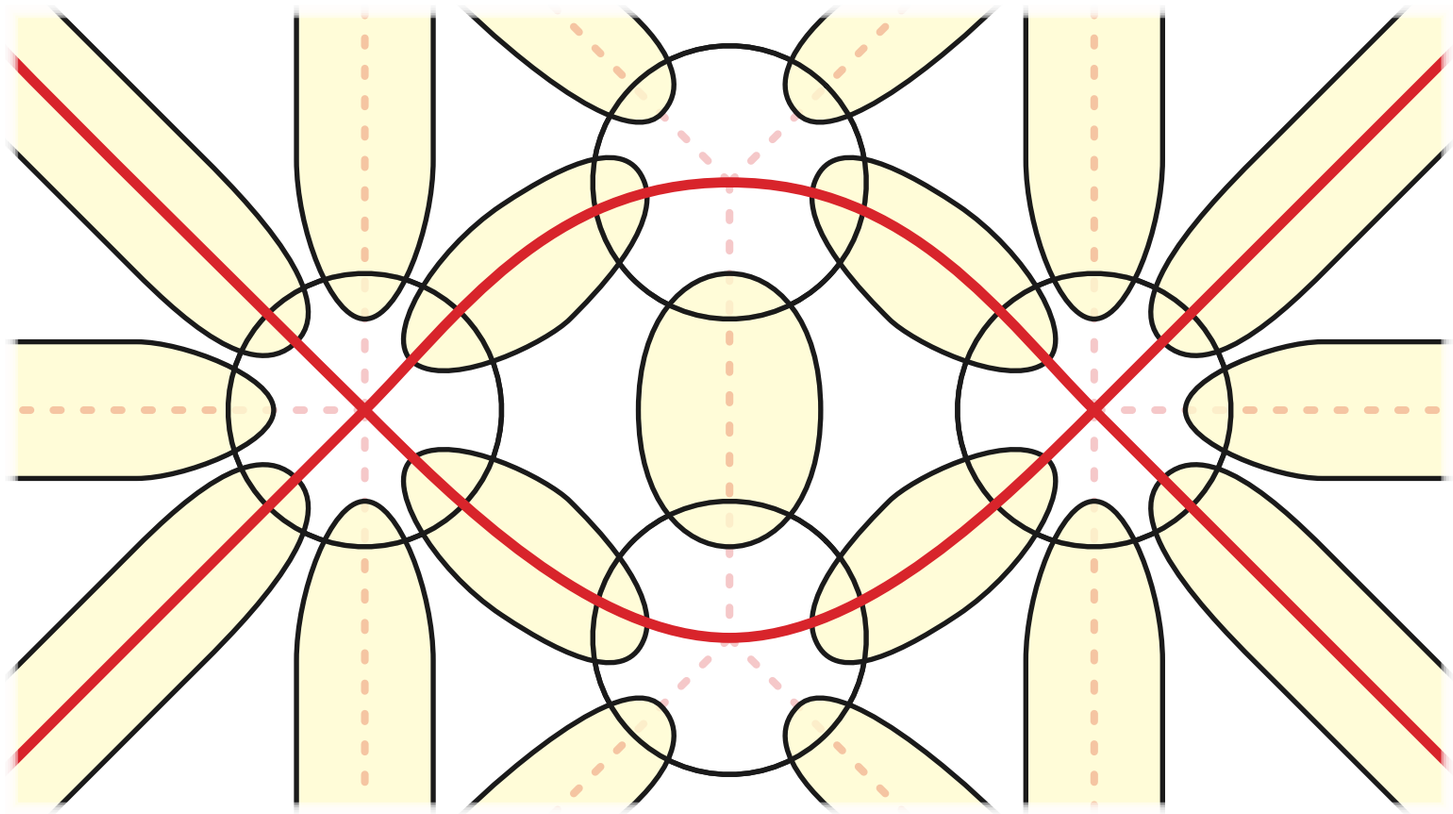


[Babson Chan '08]

- Mitchell gets this slightly wrong.

Bubble-wrapping

- Any surface immersion can be refined into the dual of a hex mesh.

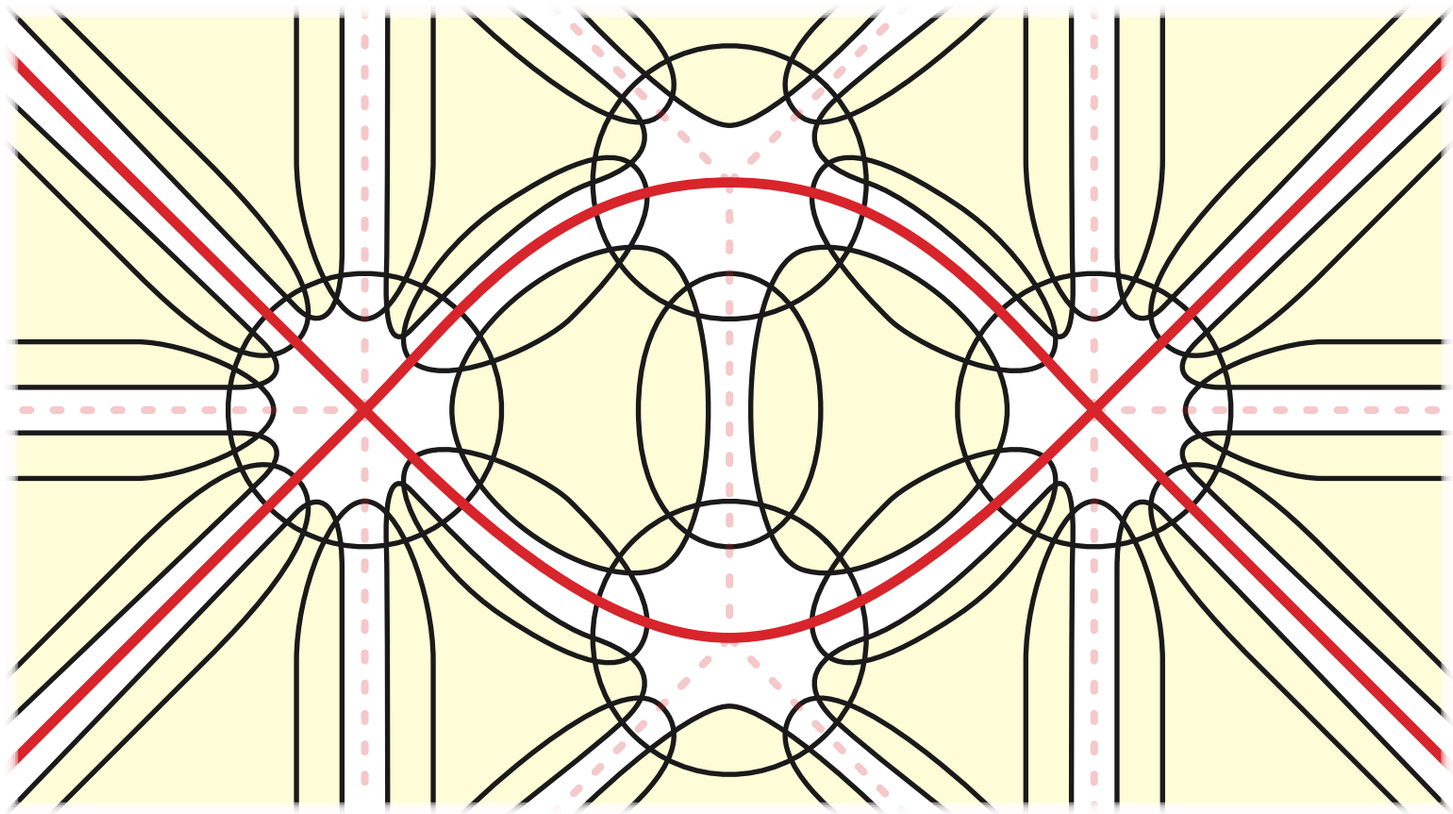


[Babson Chan '08]

- Mitchell gets this slightly wrong.

Bubble-wrapping

- Any surface immersion can be refined into the dual of a hex mesh.

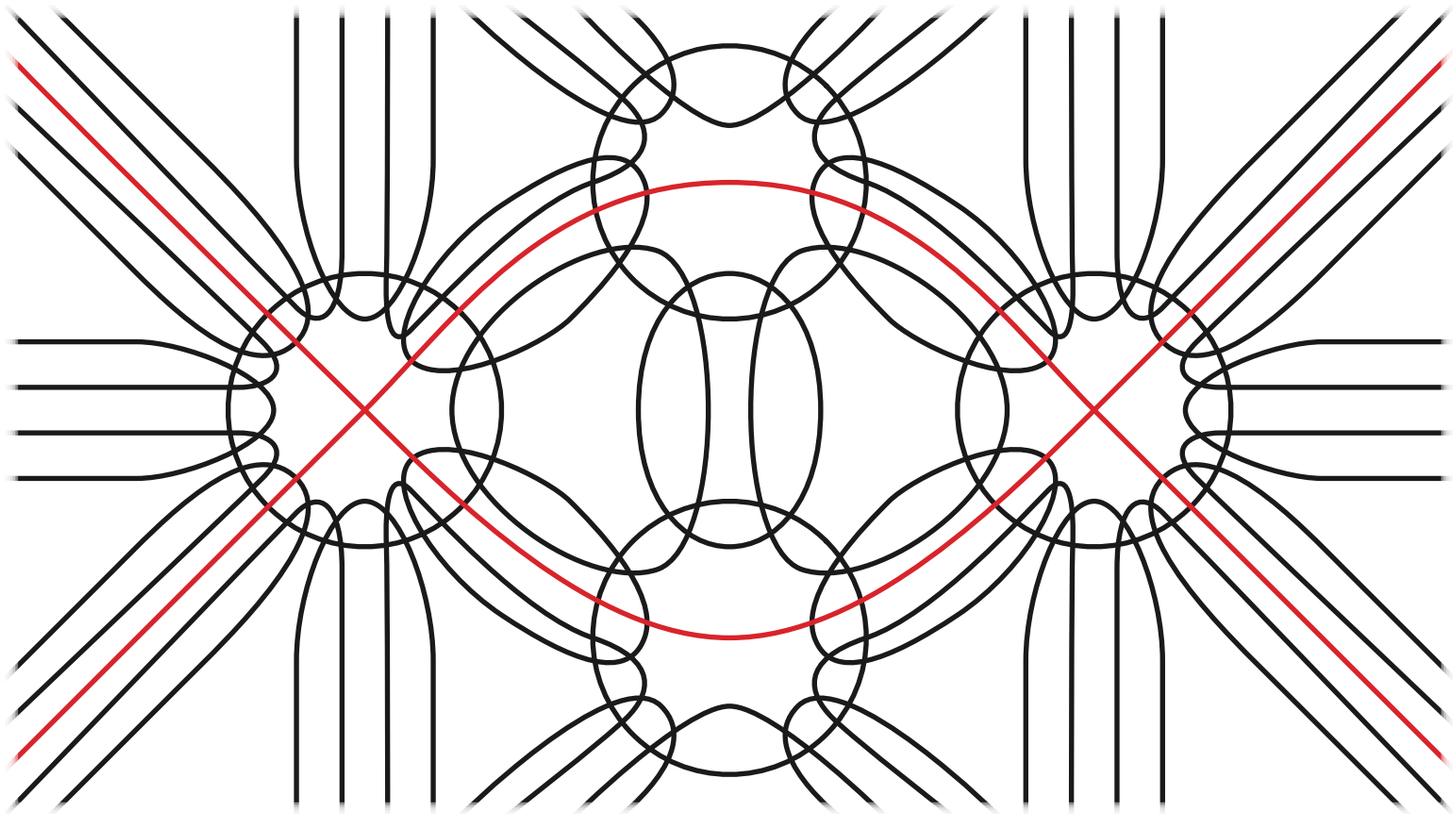


[Babson Chan '08]

- Mitchell gets this slightly wrong.

Bubble-wrapping

- Any surface immersion can be refined into the dual of a hex mesh.



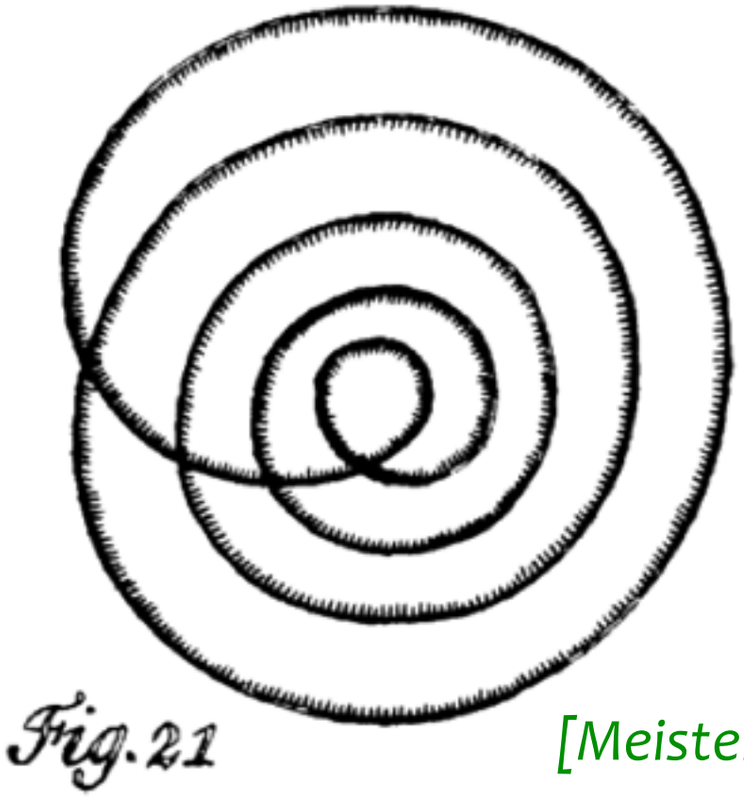
[Babson Chan '08]

- Mitchell gets this slightly wrong.

Complexity

- $O(n^2)$ moves are always enough *[Francis '69, Nowik '08]*
  $\Rightarrow O(n^2)$ hexes in the worst case

- $\Omega(n^2)$ triangle moves may be necessary *[Arnold '94, Nowik '08]*
  $\Rightarrow \Omega(n^2)$ hexes in the worst case



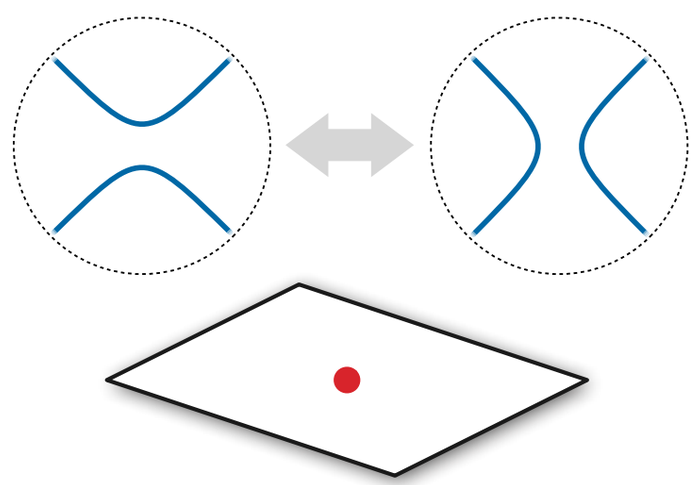
*[Meister **1769**]*

Linear complexity

- In fact, every quad mesh of the sphere with $2n$ quads is the boundary of a hex mesh of the ball with only $O(n)$ quads. *[Eppstein '99]*

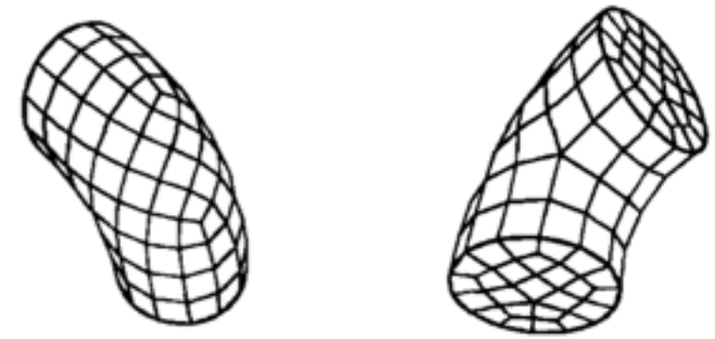
- **Eppstein's proof:** See my talk this morning.

- **Proof #2:** If we also use *saddle* moves, we need only $O(n)$ moves. *[Csikós Szűcs '95]*

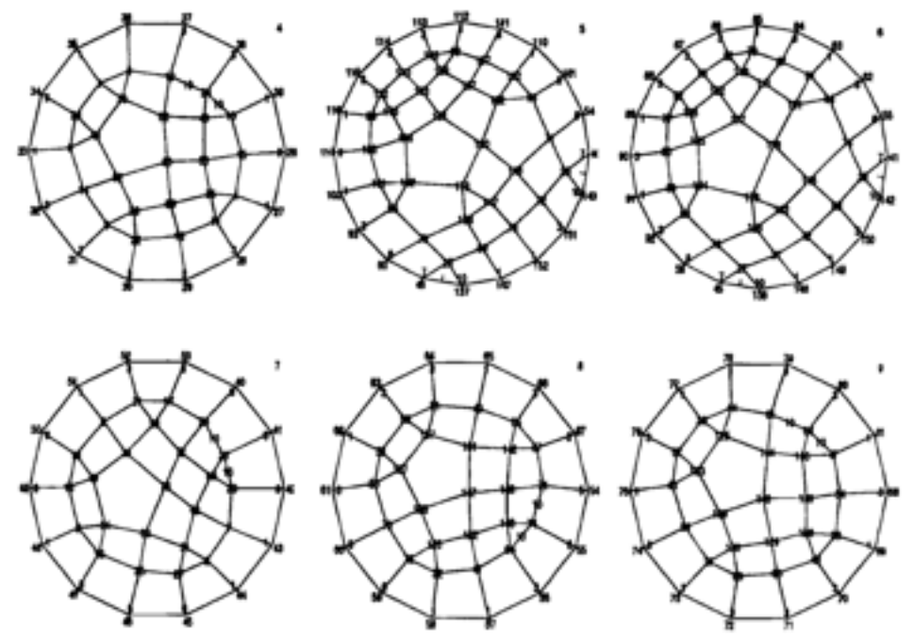


- **Proof #3:** Use cycle separators. *[Miller '86]*

Practical approaches

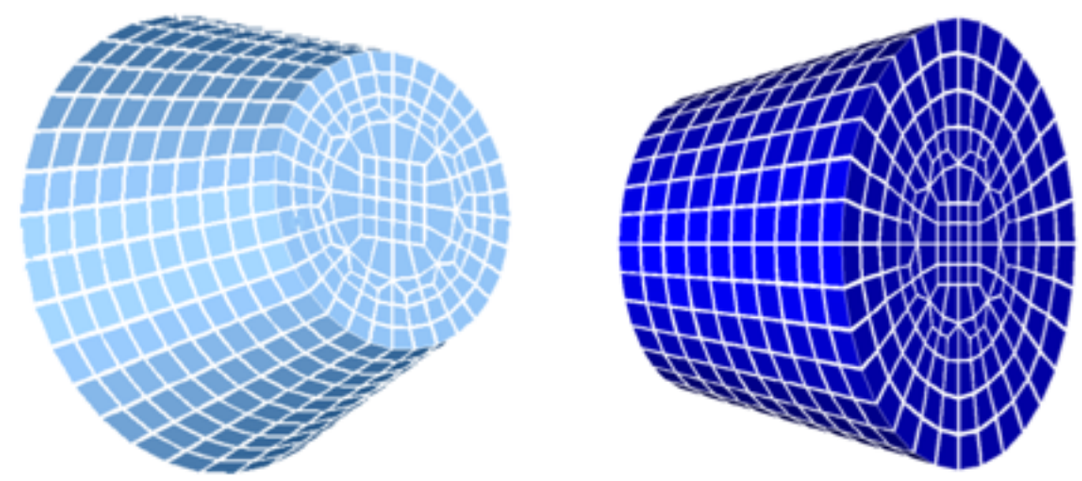


- **Whisker weaving** *[Tautges Blacker Mitchell '96, Folwell Mitchell '99]*
  - Contract dual curves inward to create dual surfaces, maintain double curves



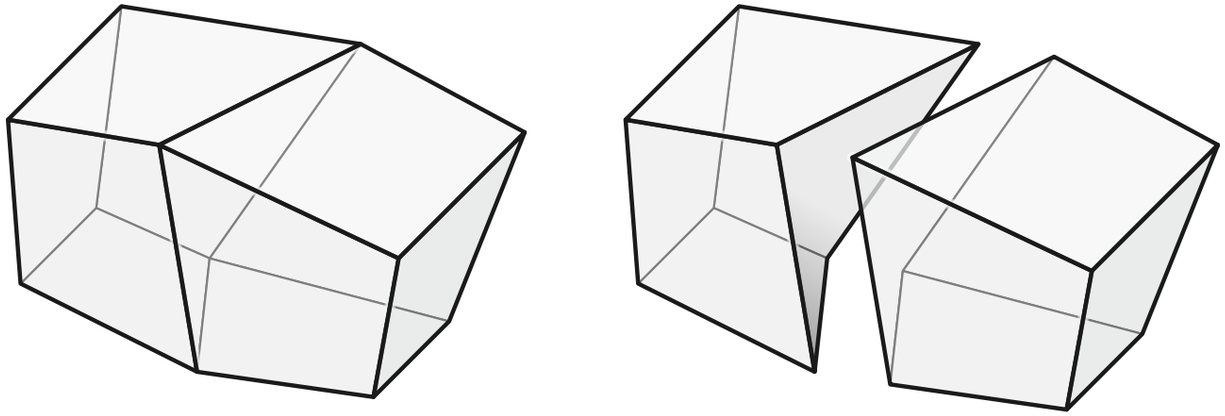
- **Cycle elimination** *[Müller-Hannemann '01]*
  - Eliminate simple dual cycles one at a time
  - In reverse: For each dual cycle, introduce a layer of hexes

- Both methods require dual curves *without self-intersections*.



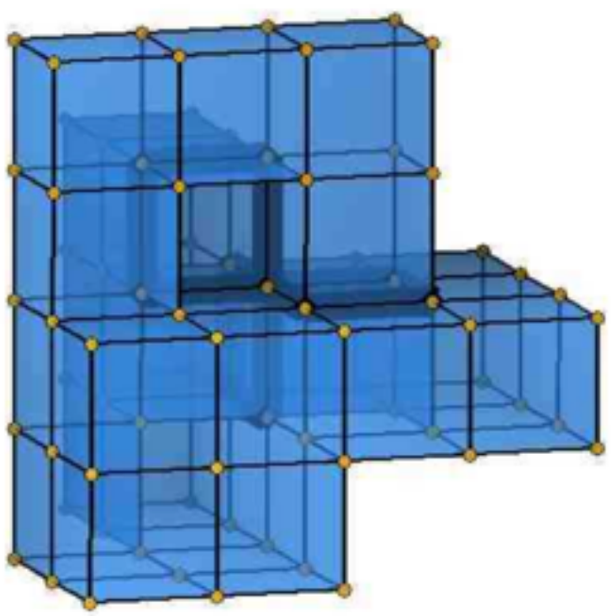
Further extensions

- Eppstein's algorithm actually works for *bipartite* surface meshes with *arbitrary topology*. *[Eppstein '99]*
- Generalizes to higher-genus surface meshes satisfying a certain topological constraint. *[Erickson '13]*
- Sufficient regular refinement gives us a mesh of *trilinear* hexes
  - ...with ***terrible*** quality, *especially* near the boundary. *[Bern Eppstein '01]*
- Sufficient regular refinement gives us a mesh of *polyhedral* hexes if and only if every *bicuboid* has a polyhedral hex mesh. *[Bern Eppstein '01]*

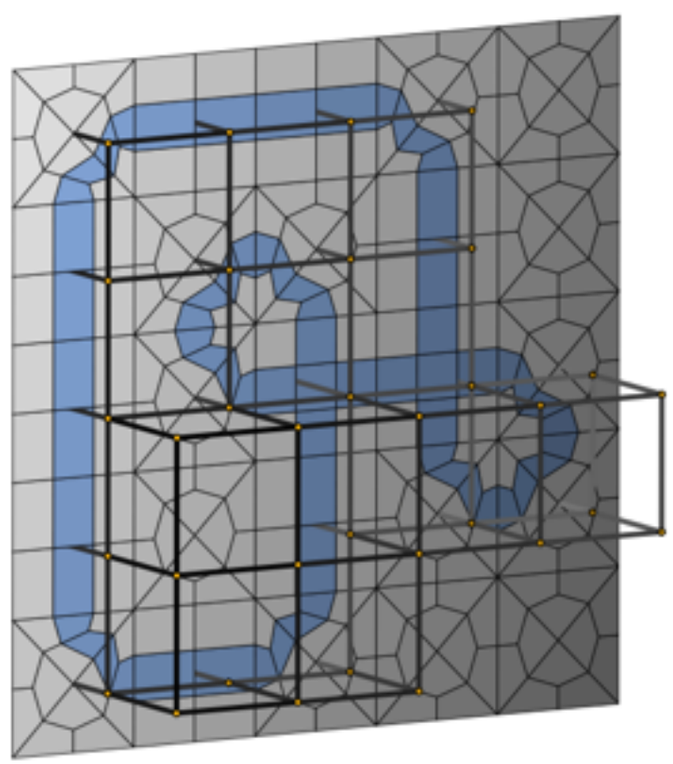


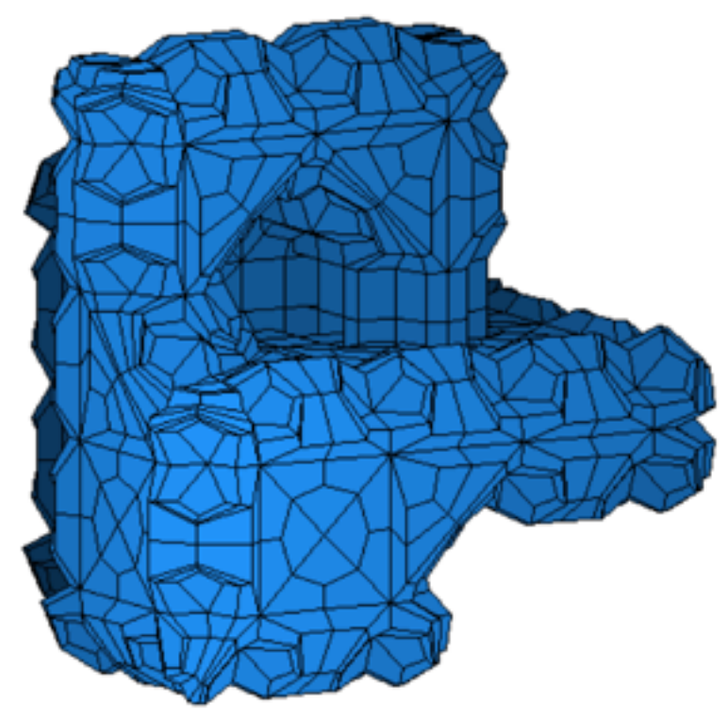
Extreme bubble-wrapping

- For *any* immersed surface Σ in $\boldsymbol{R}^3$, there is a *cubical 4-polytope* whose dual 2-skeleton contains a subdivision of Σ. *[Schwartz Ziegler '04]*
  - Construction uses a generalization of the HexHoop template *[Yamakawa Shimada '01]*



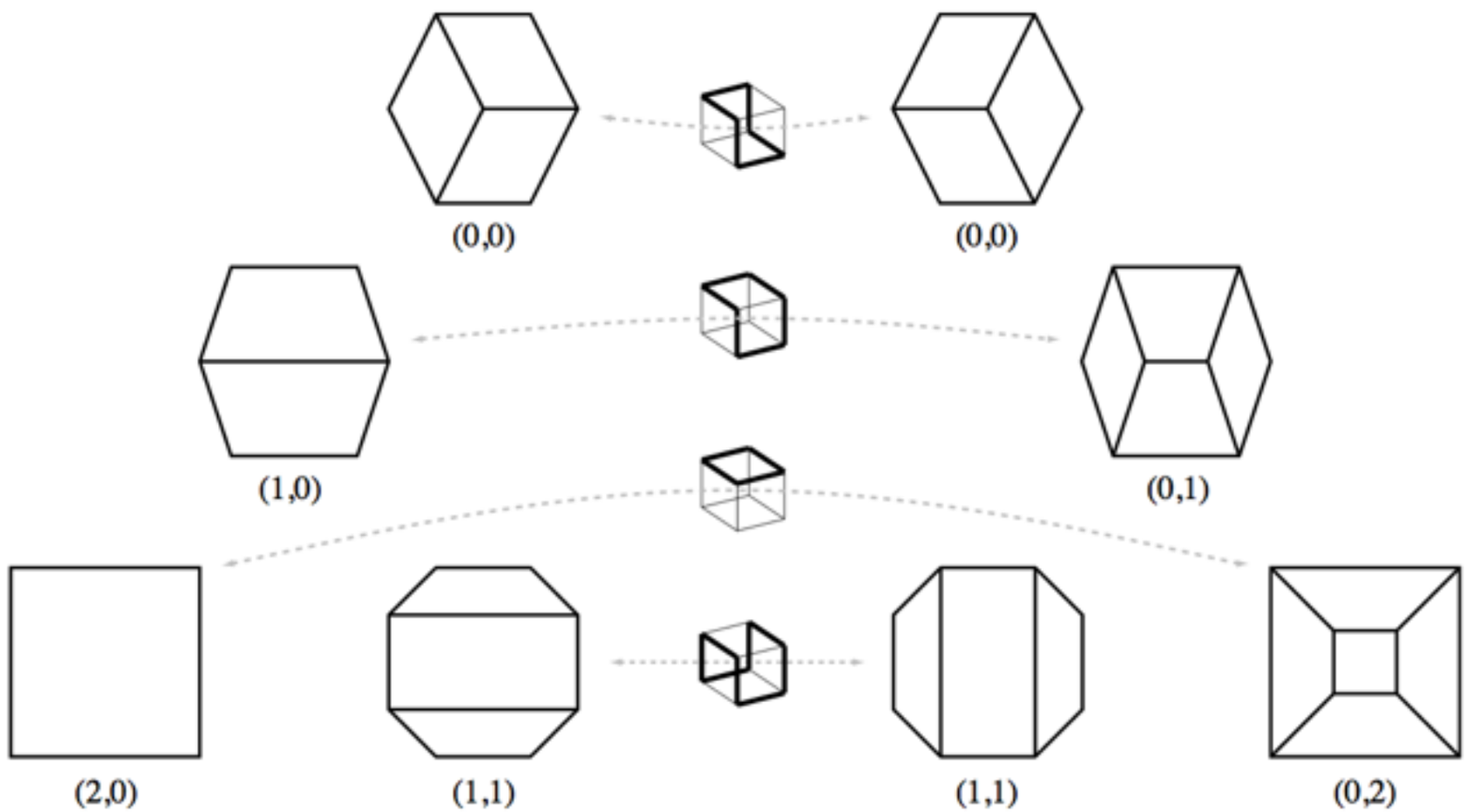
Boy's surface





Cube flips

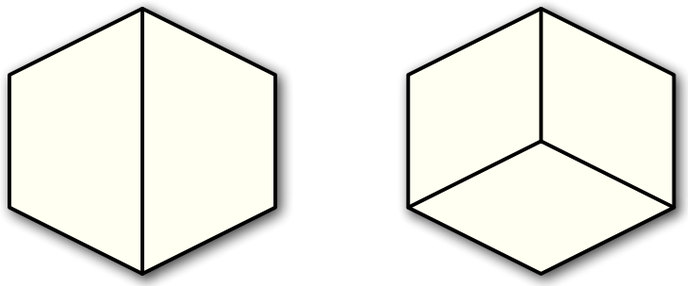
- Replace a connected subset of cube facets with its complement.
- Analog of "bistellar flips" for tetrahedral meshes. *[Alexander '30, Pachner '78]*



[Bern Eppstein '01]

Habegger's problem

- Given two cube complexes with the same underlying space, can one always be transformed into the other by cube flips? [Problem 5.13 in Kirby's *Problems in Low-Dimensional Topology*, 1995]
  - Trivially, no! Cube flips preserve parity.



- So *when* are two homeomorphic cube complexes connected by flips?

- Any two PL triangulations of the same *PL manifold* are connected by bistellar flips. *[Pachner '78, '90]*

Cubulations mod flips

- **Conjecture:** Two cubulations of the same *manifold* are connected by cube flips if and only their dual surface immersions are *cobordant*. *[Funar '99]*.

- **Theorem:** Two quad meshes of the same *2-manifold* are equivalent if and only if they have
  - the same number of quads *mod 2*,
  - and *homologous* dual curves. *[Funar '07]*

Cubulations mod flips

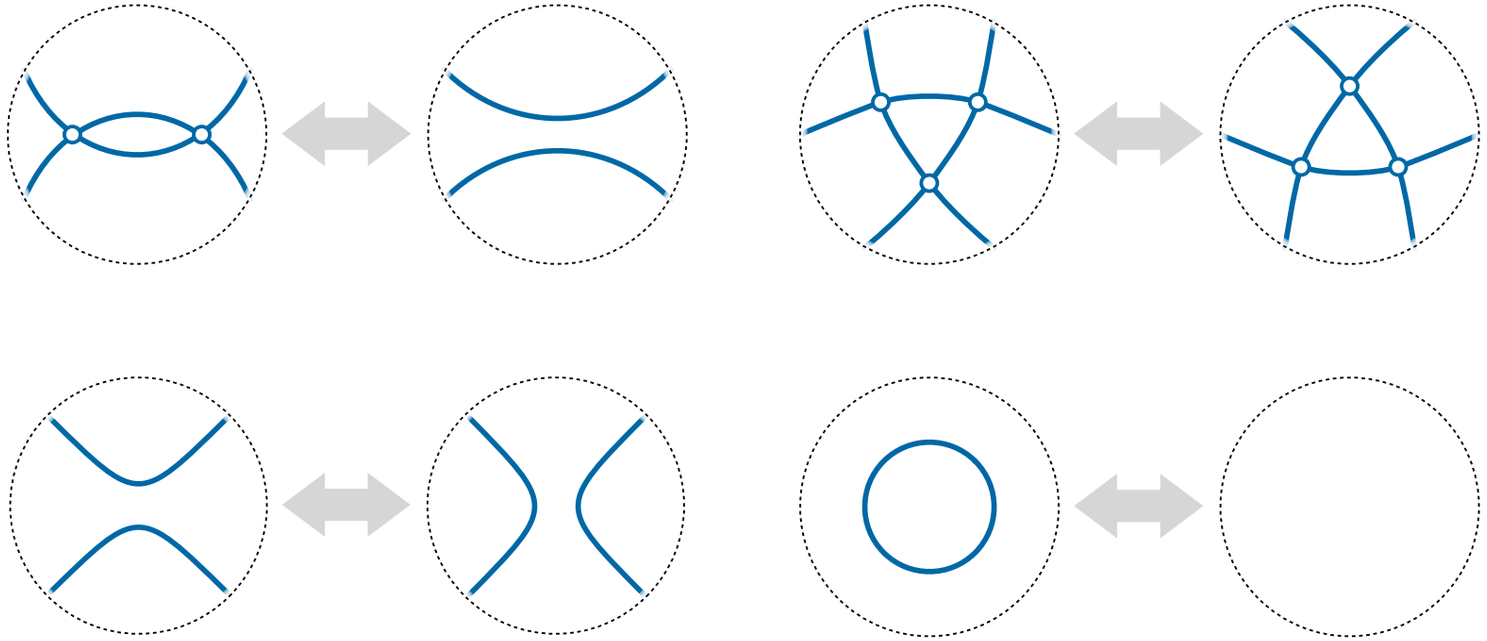
- **Conjecture:** Two cubulations of the same *manifold* are connected by cube flips if and only their dual surface immersions are *cobordant*. *[Funar '99]*.

- **Theorem:** Two quad meshes of the same *2-manifold* are equivalent if and only if they have
  - the same number of quads *mod 2*,
  - and *homologous* dual curves. *[Funar '07]*

This was the key inspiration for my SOCG result.

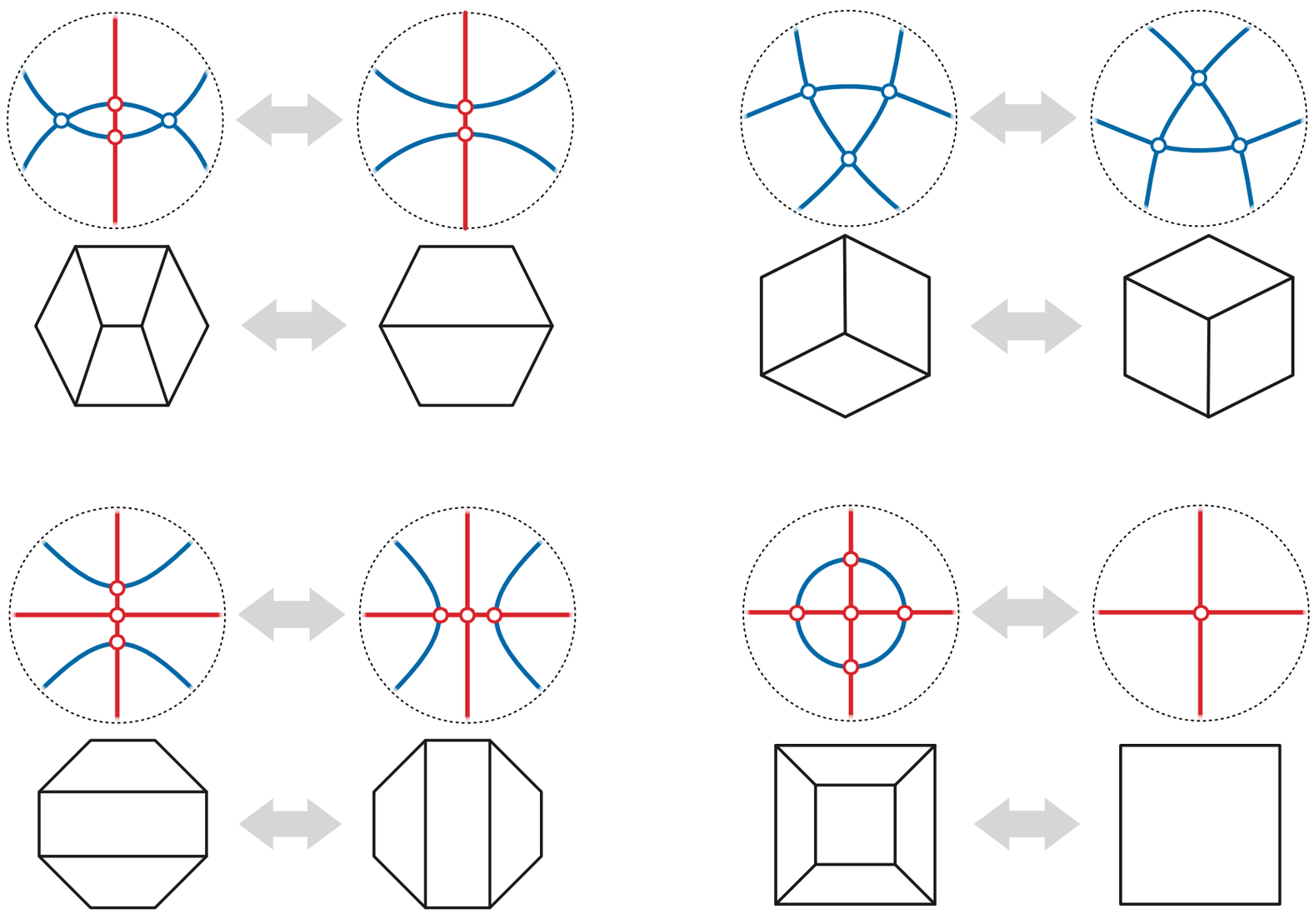
Elementary moves

- **Lemma:** Two immersed curves with the same parity are homologous iff one can be transformed to the other using these elementary moves:



Bubble-wrapped moves

- **Lemma:** After suitable refinement, each elementary move can be executed by a finite sequence of cube flips. *[Funar '07, Bern Eppstein '01]*



Summary

- With boundary refinement:
  - Hex meshing is *no harder than tet meshing*.
  - **Good** hex meshing, on the other hand, is a black art.

- Without boundary refinement:
  - Topological hex meshing is *solved*...
    - ...except for tight bounds on worst-case mesh complexity.
  - Trilinear hex meshing is *"solved"*...
    - ...but ***terrible*** quality, especially near boundary; ***no*** bounds on mesh complexity
  - Polyhedral hex meshing reduces to bicuboids
  - ***No*** quality guarantees of any kind.

The open problem

The open problem

- Define *useful* but *general* constraints on the input geometry.
  - Polyhedra with all angles $\geq 90^\circ$?
  - Smooth surfaces?

The open problem

- Define *useful* but *general* constraints on the input geometry.
  - Polyhedra with all angles $\geq 90°$?
  - Smooth surfaces?

- Define *useful* quality constraints for trilinear hexahedral meshes.
  - All (scaled) Jacobian determinants ≥ 0.1?
  - All Jacobians *close* to (scaled) orthogonal? (What does *close* mean?)
  - In every hex, *similar* Jacobians at all 8 vertices? (What does *similar* mean?)
  - Number of hexes $= O(\int 1/\ \mathrm{lfs}^3)$?

The open problem

- Define *useful* but *general* constraints on the input geometry.
  - Polyhedra with all angles $\geq 90°$?
  - Smooth surfaces?
- Define *useful* quality constraints for trilinear hexahedral meshes.
  - All (scaled) Jacobian determinants ≥ 0.1?
  - All Jacobians *close* to (scaled) orthogonal? (What does *close* mean?)
  - In every hex, *similar* Jacobians at all 8 vertices? (What does *similar* mean?)
  - Number of hexes $= O(\int 1/\ \mathrm{lfs}^3)$?
- Given an *arbitrary* volume meeting the input constraints, *provably* compute a hex mesh that *provably* meets the output constraints.

The open problem

- Define *useful* but *general* constraints on the input geometry.
  - Polyhedra with all angles $\geq 90°$?
  - Smooth surfaces?
- Define *useful* quality constraints for trilinear hexahedral meshes.
  - All (scaled) Jacobian determinants ≥ 0.1?
  - All Jacobians *close* to (scaled) orthogonal? (What does *close* mean?)
  - In every hex, *similar* Jacobians at all 8 vertices? (What does *similar* mean?)
  - Number of hexes $= O(\int 1/\ \mathrm{lfs}^3)$?
- Given an *arbitrary* volume meeting the input constraints, *provably* compute a hex mesh that *provably* meets the output constraints.
- ...and make it work in practice!

I hate meshes.
I cannot believe how hard this is.
Geometry is hard.

— David Baraff

Senior Research Scientist
Pixar Animation Studios