Brief (Pre-)History of Computational Topology

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SoCG 2015

Workshop on Computational Topology

June 22, 2015

Brief (Pre-)History of
One Small Part of
Computational Topology

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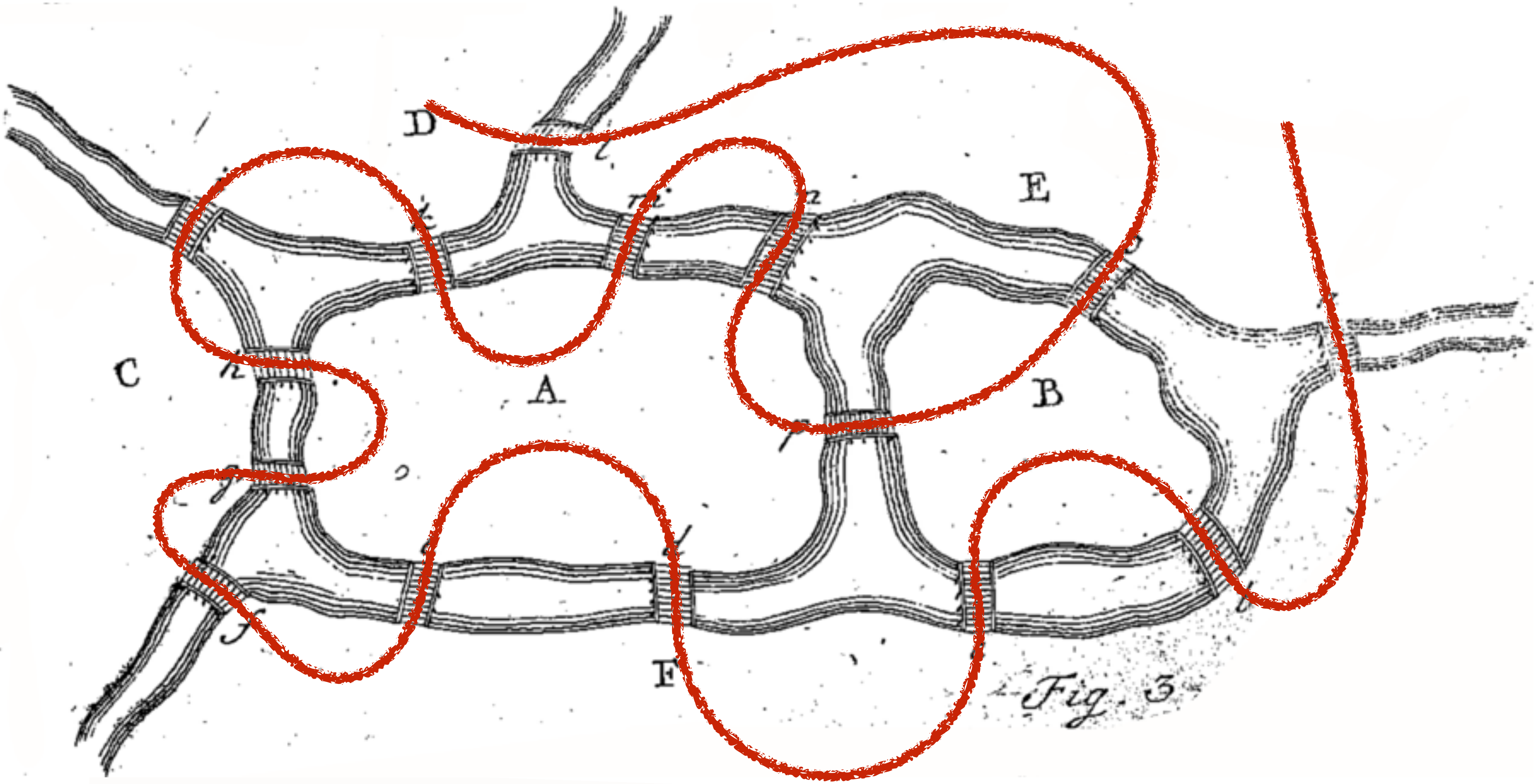
Ponam ſemper pro numero angulorum ſolidorum α & pro numero facierum φ. Aggregatum ex omnibus angulis planis eſt $4\alpha - 8$, & numerus φ eſt $2\alpha - 4$, ſi numerentur tot facies quot poſſunt eſſe triangula. Numerus item angulorum planorum eſt $6\alpha - 12$, numerando ſcilicet vnum angulum pro tertiâ parte duorum rectorum. Nunc ſi ponam 3α pro tribus angulis planis qui ad minimum requiruntur vt componant vnum angulum angulorum ſolidorum, ſuperſunt $3\alpha - 12$, quæ ſumma addi debet ſingulis angulis ſolidis juxta tenorem quæſtionis, ita vt æqualiter omni ex parte diffundantur. Numerus verorum angulorum planorum eſt $2\varphi + 2\alpha - 4$, qui non debet eſſe major quàm $6\alpha - 12$; ſed ſi minor eſt, exceſſus erit $+ 4\alpha - 8 - 2\varphi$.



Cursus autem ita fieri poterit E *a* F *b* B *c* F *d* A *e* F *f* C *g* A *h* C *i* D *k* A *m* E *n* A *p* B *o* E *l* D vbi inter litteras maiusculas pontes simul collocaui, per quos fit transitus.

$$
\varphi^{\mathrm{IV}}\left\{\begin{array}{l|l}
 & (a) \\
a & \\
 & (ab)\,(c) \\
bc & \\
 & (bde)\,(cf) \\
def & \\
 & (eh)\,(dfg) \\
gh & \\
 & (hk)\,(gi)\,(l) \\
ikl & \\
 & (ikm)\,(ln) \\
mn & \\
 & (mno) \\
o & \\
 & (o)
\end{array}\right. ,
\qquad
\varphi^{\mathrm{V}}\left\{\begin{array}{l|l}
 & (a) \\
a & \\
 & (abc) \\
bc & \\
 & (bd)\,(cef) \\
def & \\
 & (dgh)\,(ei)\,(fk) \\
ghik & \\
 & (gl)\,(him)\,(kn) \\
lmn & \\
 & (lmo)\,(np) \\
op & \\
 & (opq) \\
q & \\
 & (q)
\end{array}\right. ,
$$

womit man die nachstehenden beiden Figuren vergleichen mag:

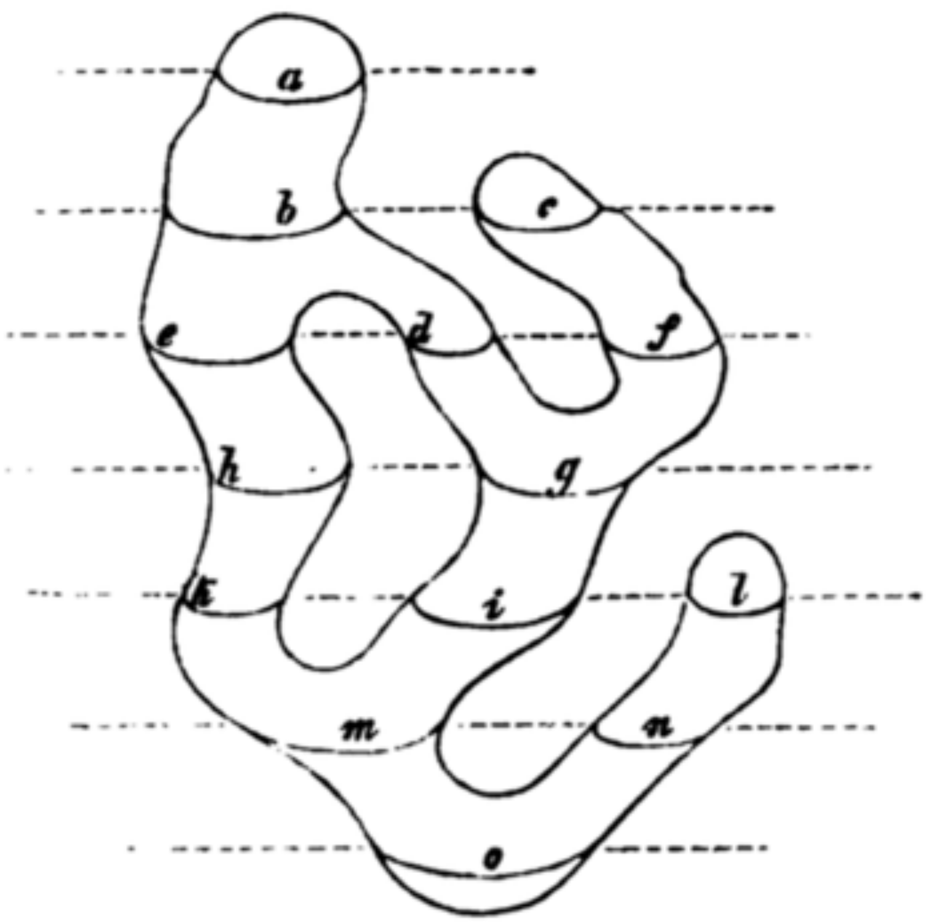


Fig. 6.

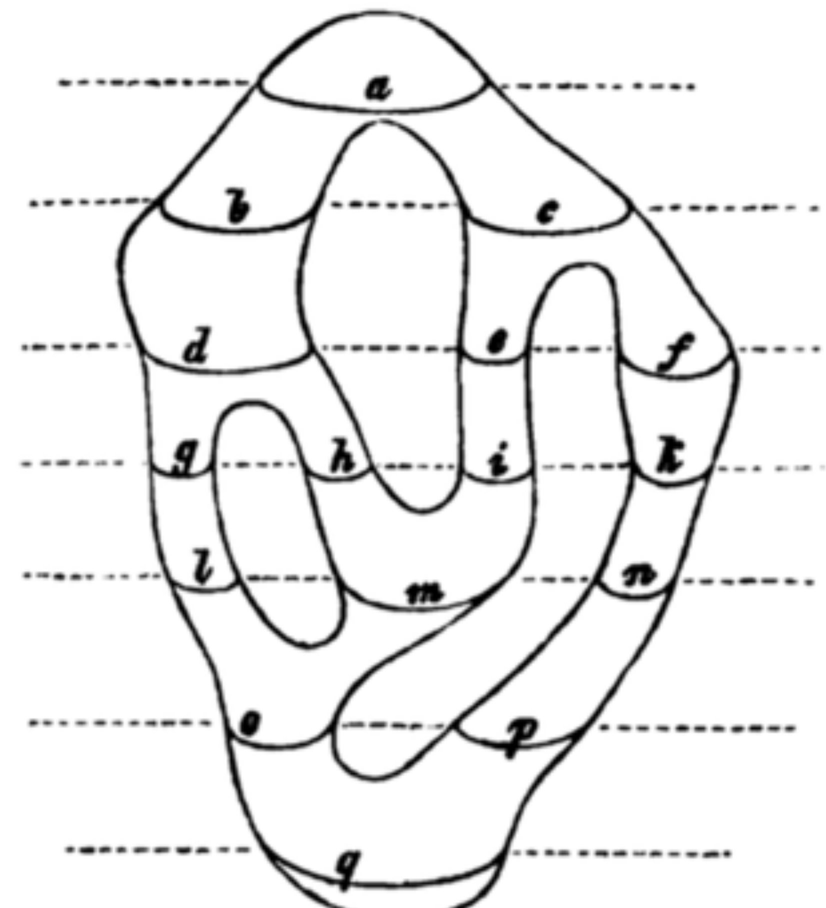
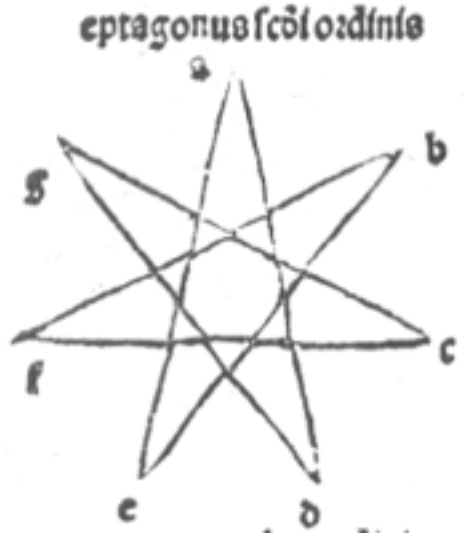
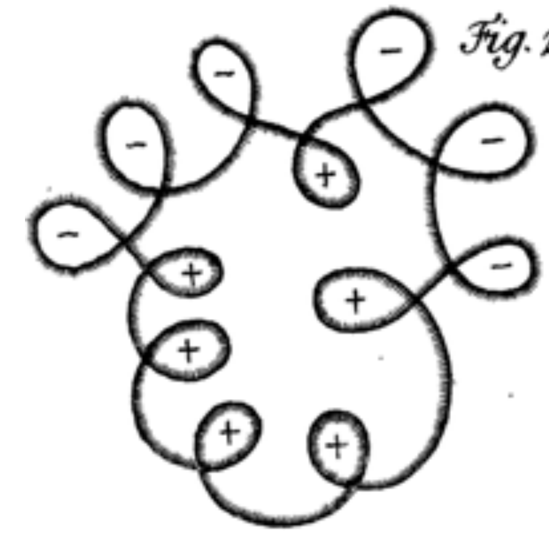
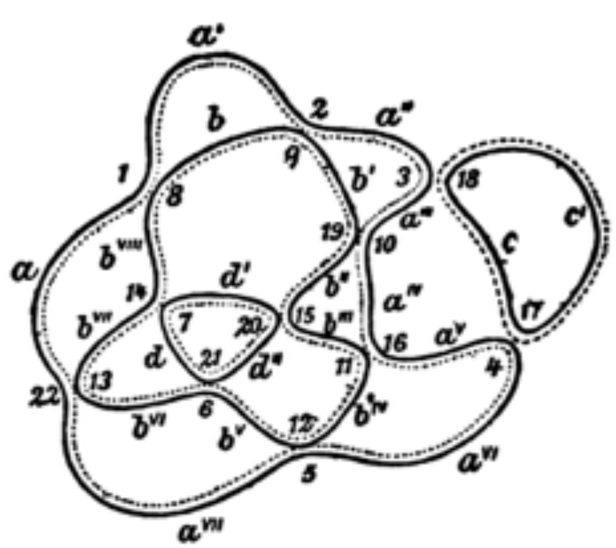


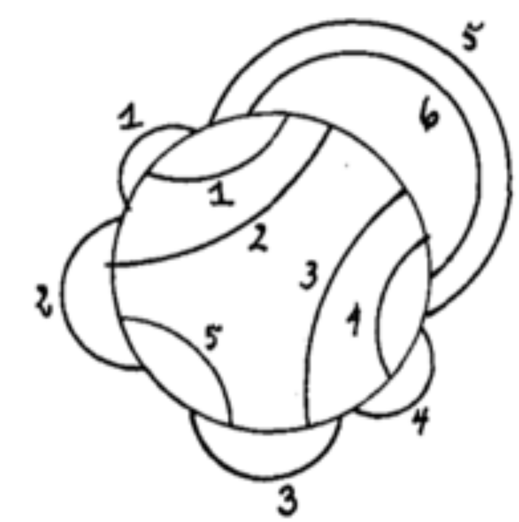
Fig. 7.





Polygons and Curves





Thomas Bradwardine (c.1290–1349)

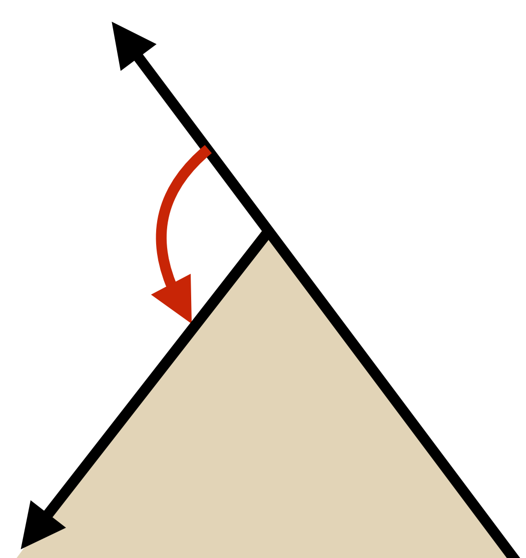
Geometria Speculativa (c. 1320)

- The interior angles of a pentagram total two right angles.
- Each additional vertex adds two right angles.
- Increasing the “order” removes four right angles.

Geometria Speculativa (c. 1320)

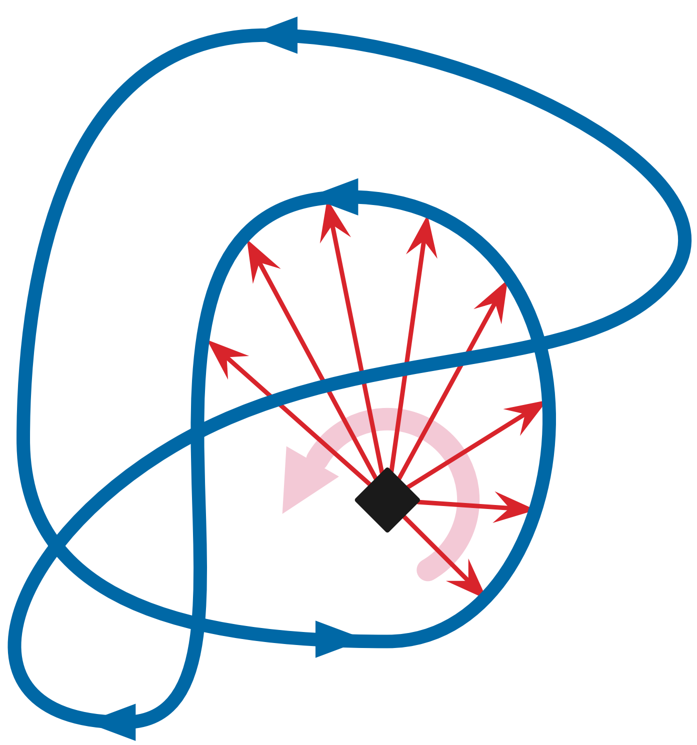
- The interior angles of a pentagram total two right angles.
- Each additional vertex adds two right angles.
- Increasing the "order" removes four right angles.

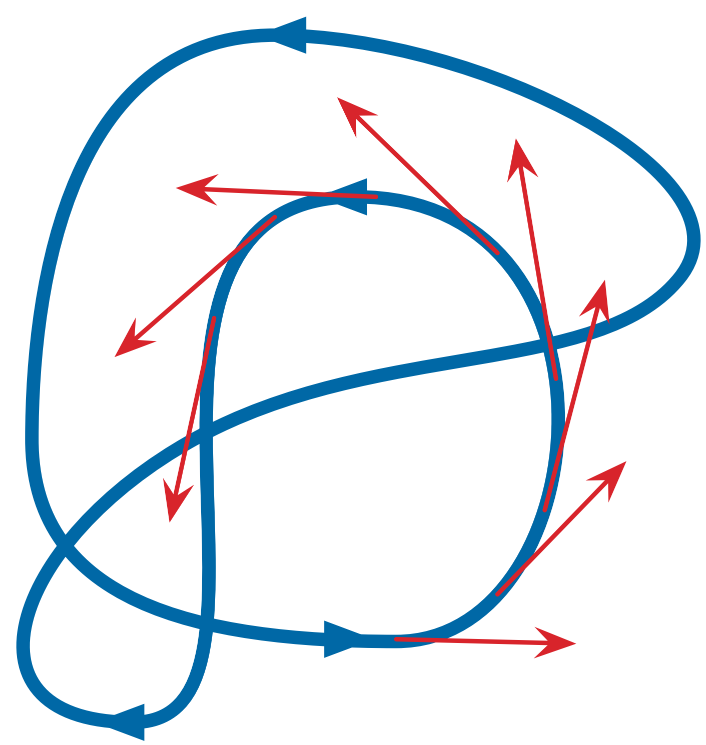
- In modern language:
  The exterior angles of a regular $\{p/q\}$-polygon sum to $2\pi q$.



Winding and rotation numbers

- Winding number = # times a curve winds around a point
- Rotation number= # turns made by tangent vector
- For polygons, rotation number = sum of external angles / 2π


wind(C, p) = 1


turn(C) = 0

Albrecht Ludwig Friedrich Meister (1724–1788)

M. ALBERT. LVDOV. FRID. MEISTER

GENERALIA DE GENESI FIGVRARVM

PLANARVM,

ET

INDE PENDENTIBVS EARVM

AFFECTIONIBVS

D. VI. IAN. CIϽ IϽCCLXX.

Meister's *Generalia* (1770)

- First to consider ***arbitrary*** polygons and (regular) curves

Fig. 26

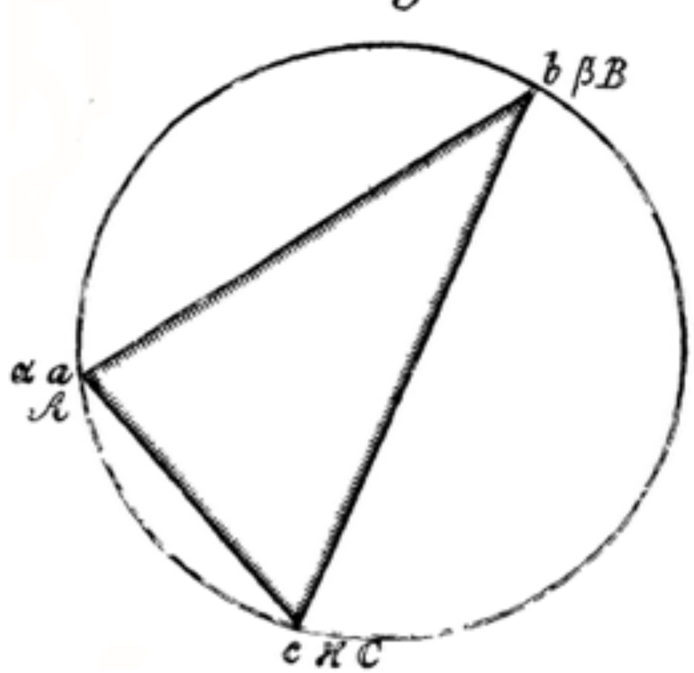


Fig. 34

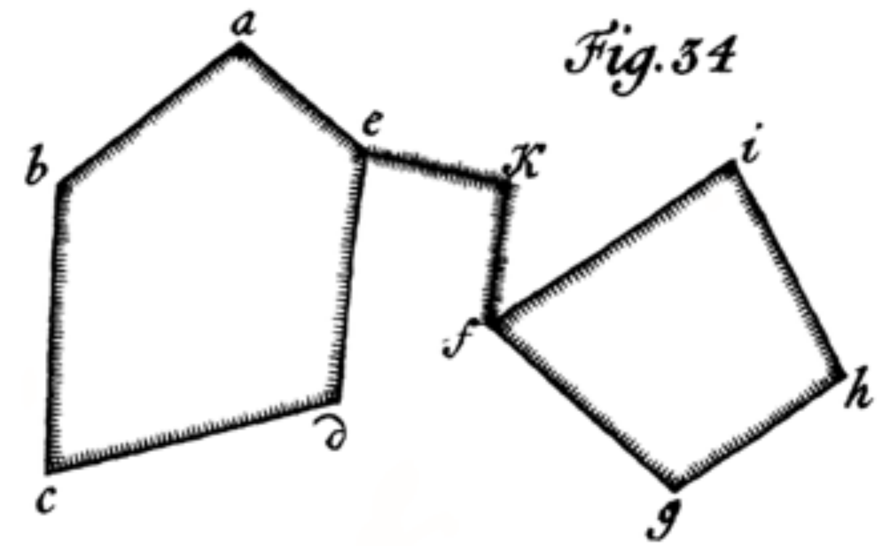


Fig. 3

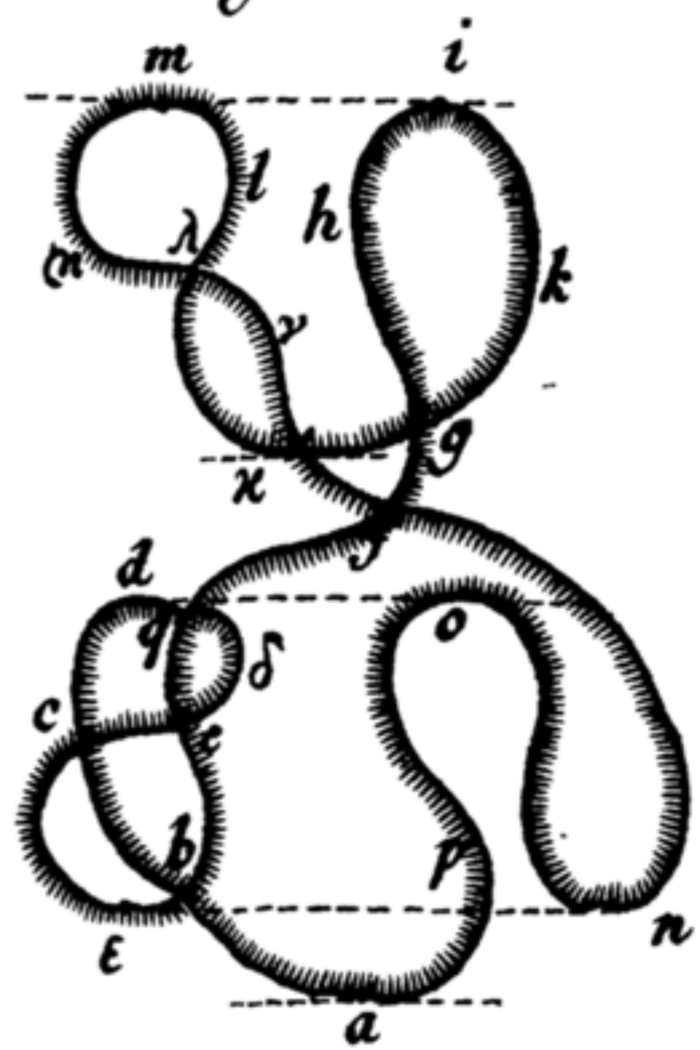


Fig. 35

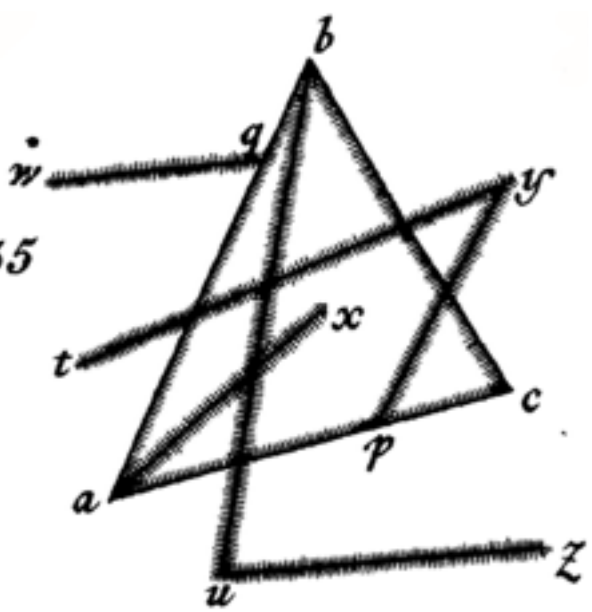
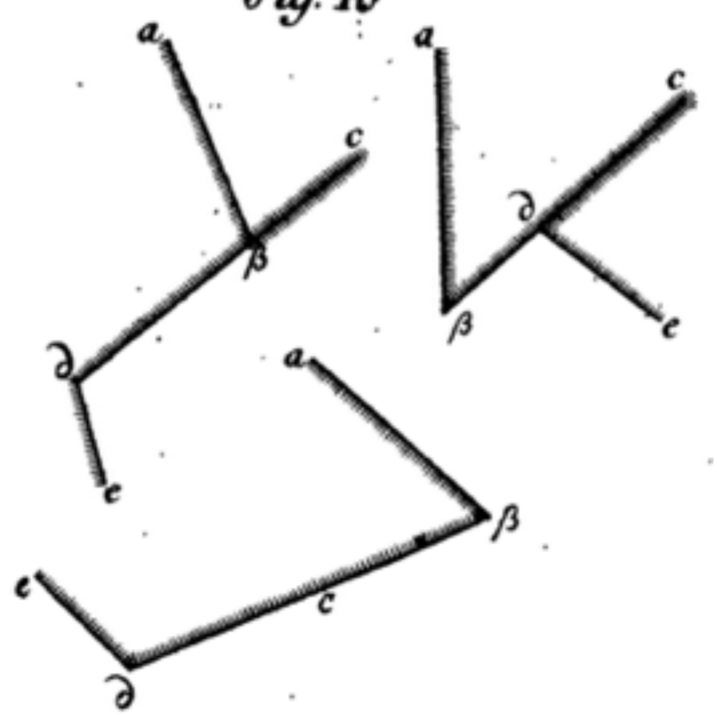
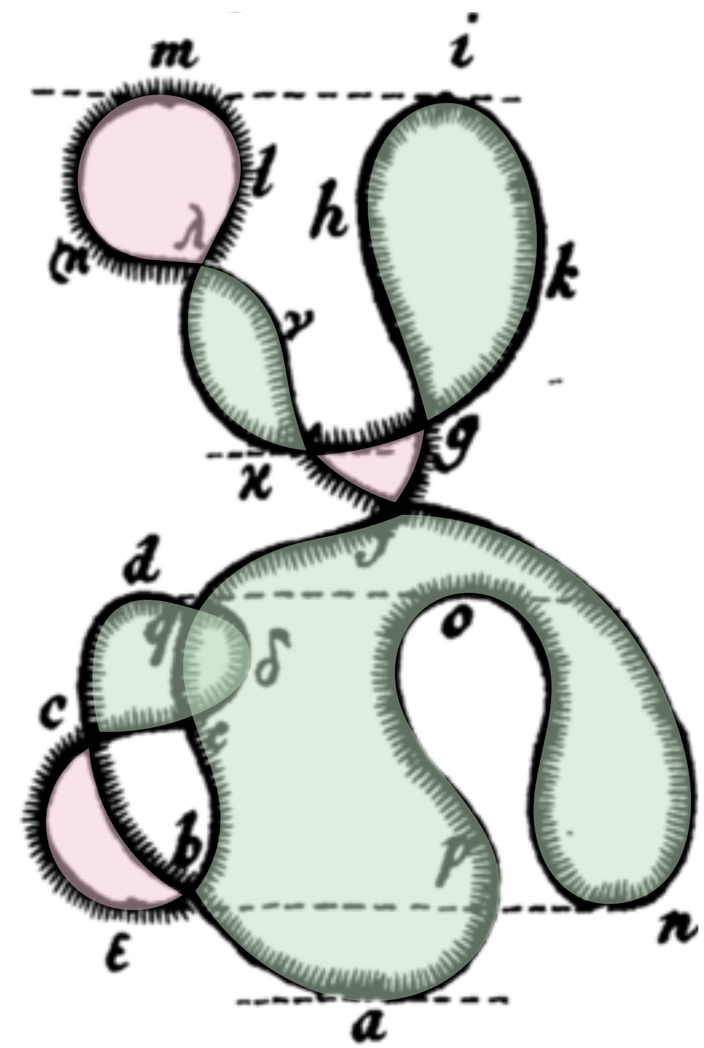


Fig. 40



Definition of signed area

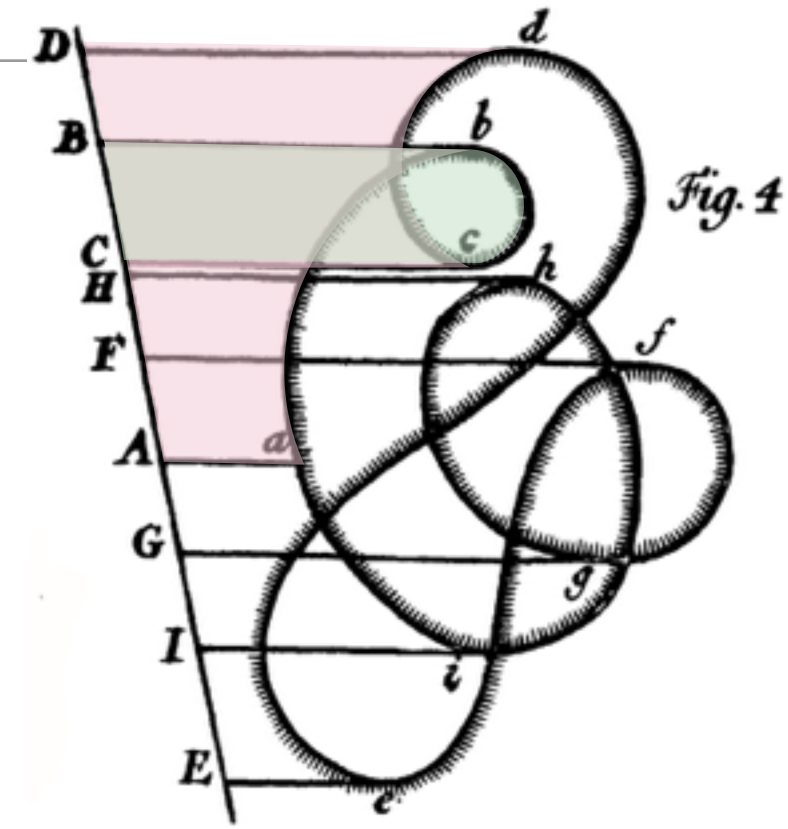
- Split the curve into *simple* loops at crossing points
- Add area of positive loops. Subtract area of negative loops.
- Contribution of any region is area × winding number

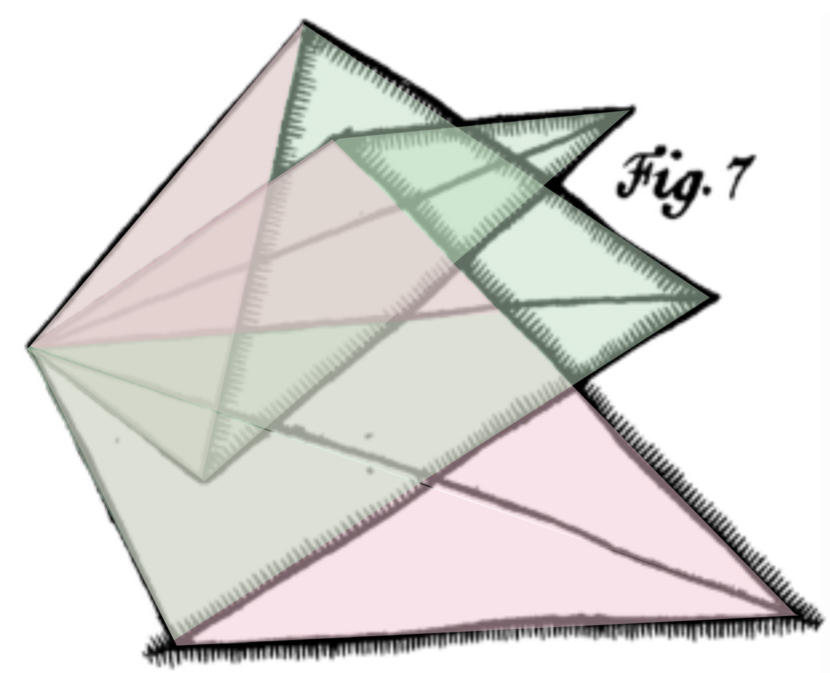


Sunt enim affirmativae + ghikg + κλνkκ + ceδqdc + abeqfnopa, ita ut pars eqδe bis cenſeatur; negativae vero — lmμλl — kgf — bεcb.

Meister's "Shoelace algorithm"

- Computing the signed area of a curve
  - Split curve at points with horizontal tangents
  - Measure area between each curve segment and a line
  - Subtract area between Aa and Bb,
    add area between Bb and Cc,
    subtract area between Cc and Bb,
    and so on
- Signed area of a polygon = Sum of signed triangle areas





Meister: Rotation number

- As sums of internal angles (like Bradwardine)
- Fig. 17: Moving e to ε doesn’t change the sum of angles
- Fig. 18: Moving c to κ changes the sum of angles by 2π.

Fig.17

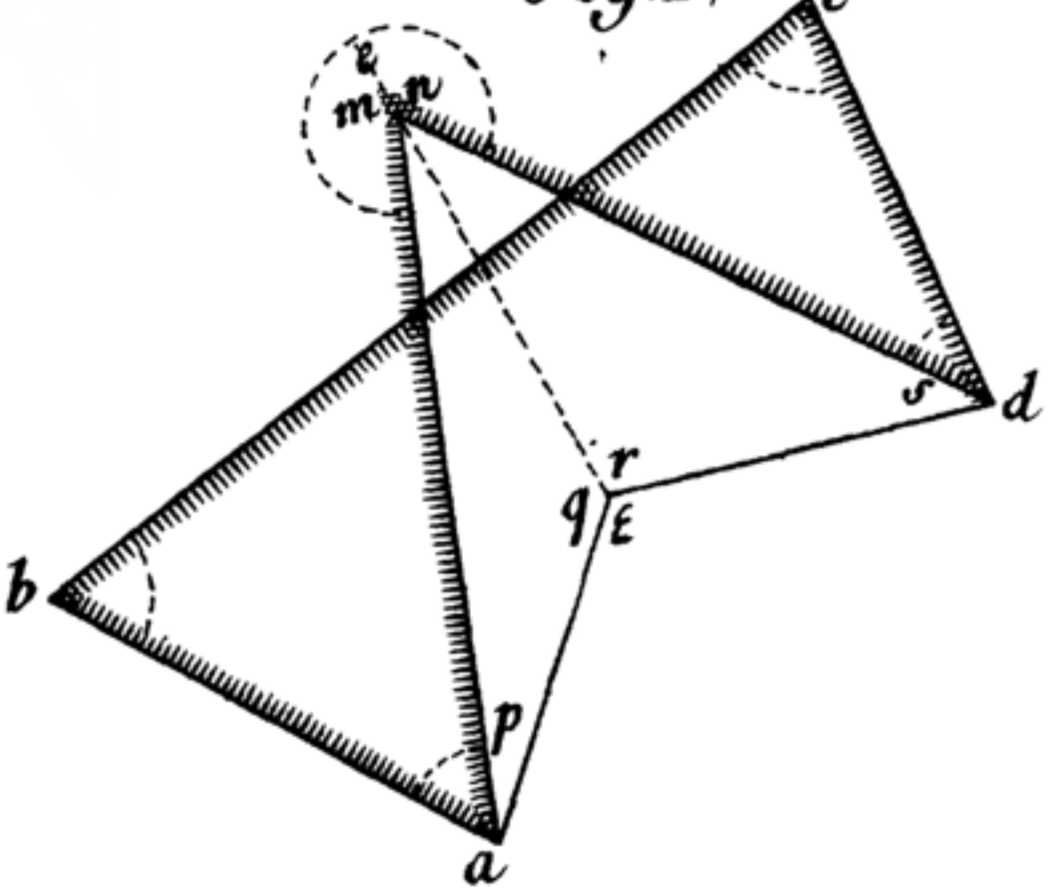
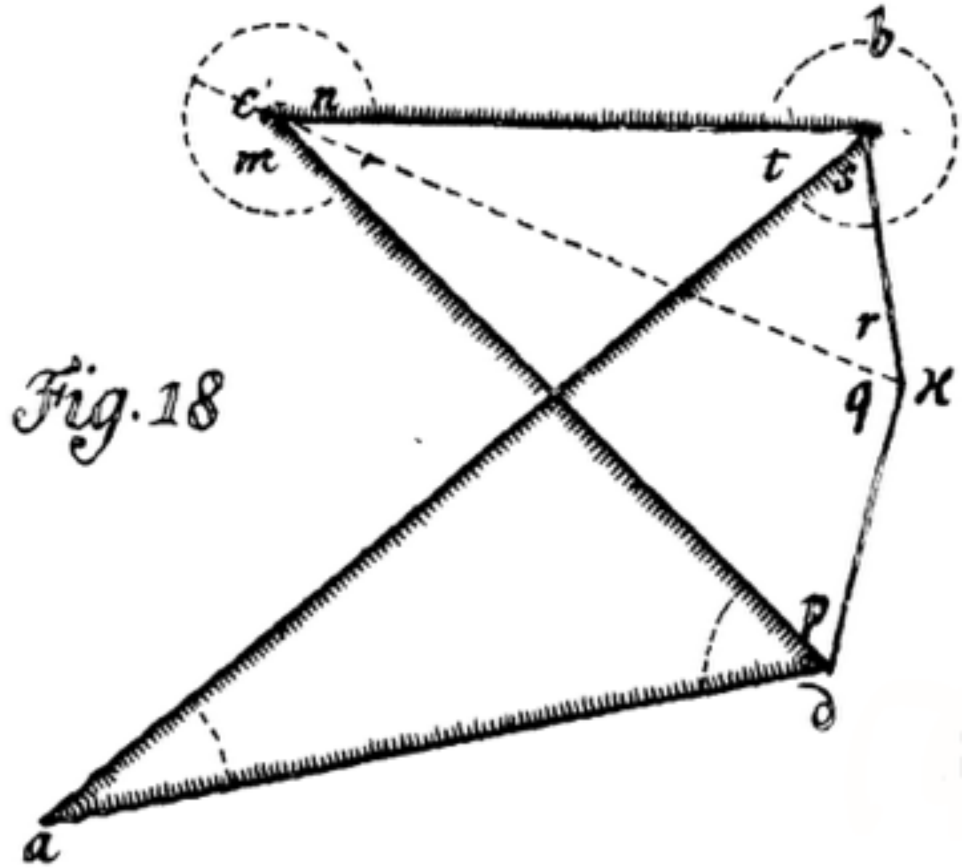
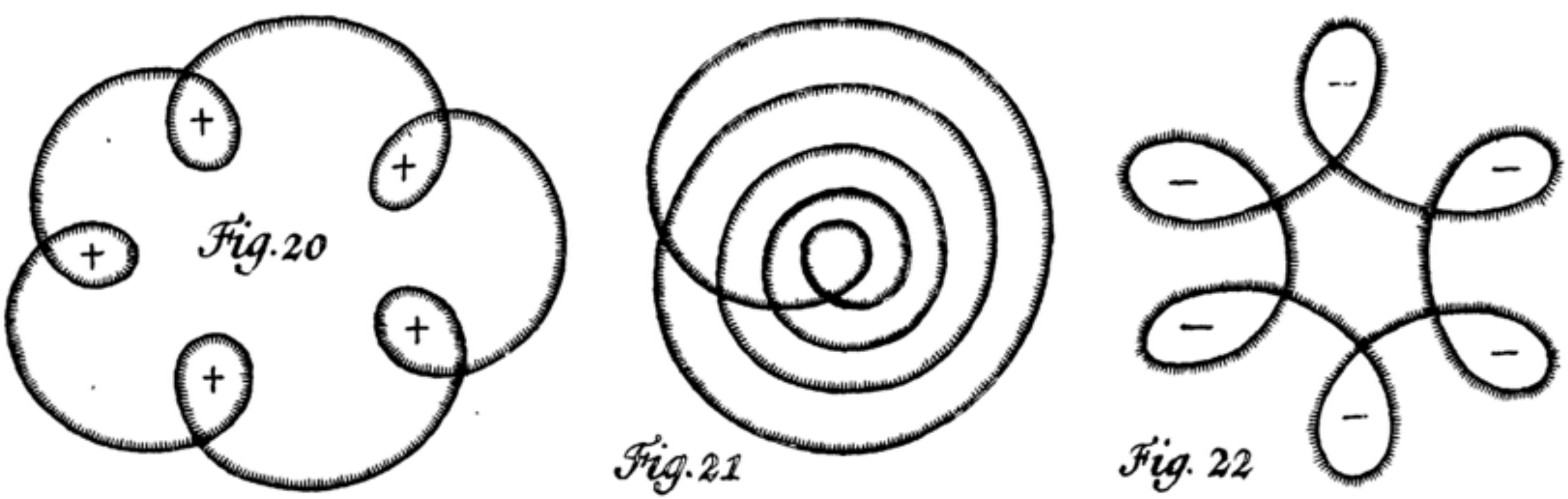


Fig.18



Meister: Rotation number

“And first it is evident that a positively complicated perimeter, for any number of complications, can be reduced to the general forms in Figures 20 and 21; and negatively complicated perimeters to the form in Figure 22 with the same number of complications, which differs from Figure 20 only as respective angles are external or internal.”



Meister: Rotation number

“Then it is clear that positive complications remove an equal number of negative ones; that if the number of both in the figure are equal, it will return to a simple figure, where the sum of the angles is determined by the number of edges in the usual manner.”

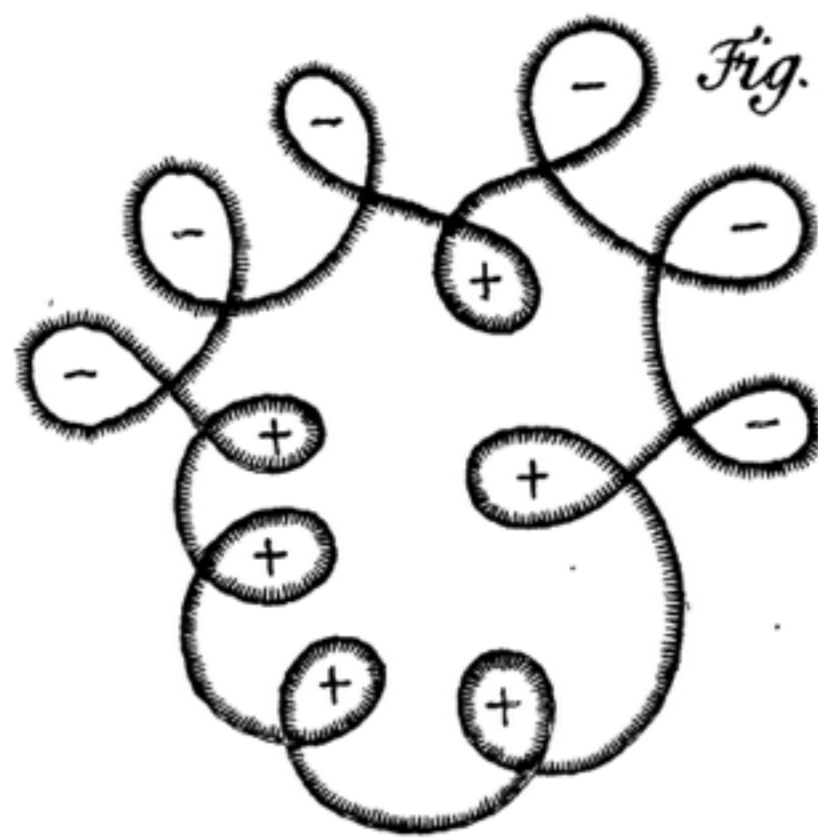
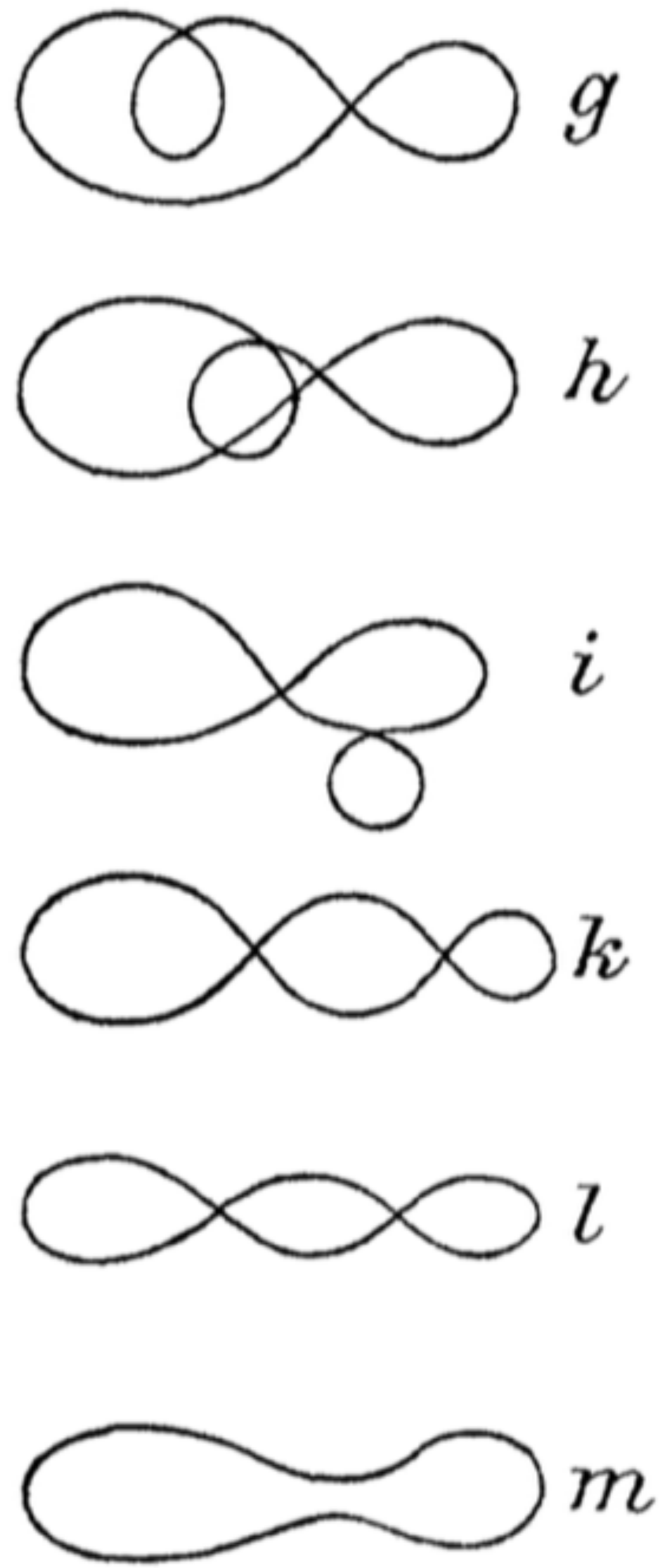


Fig. 23

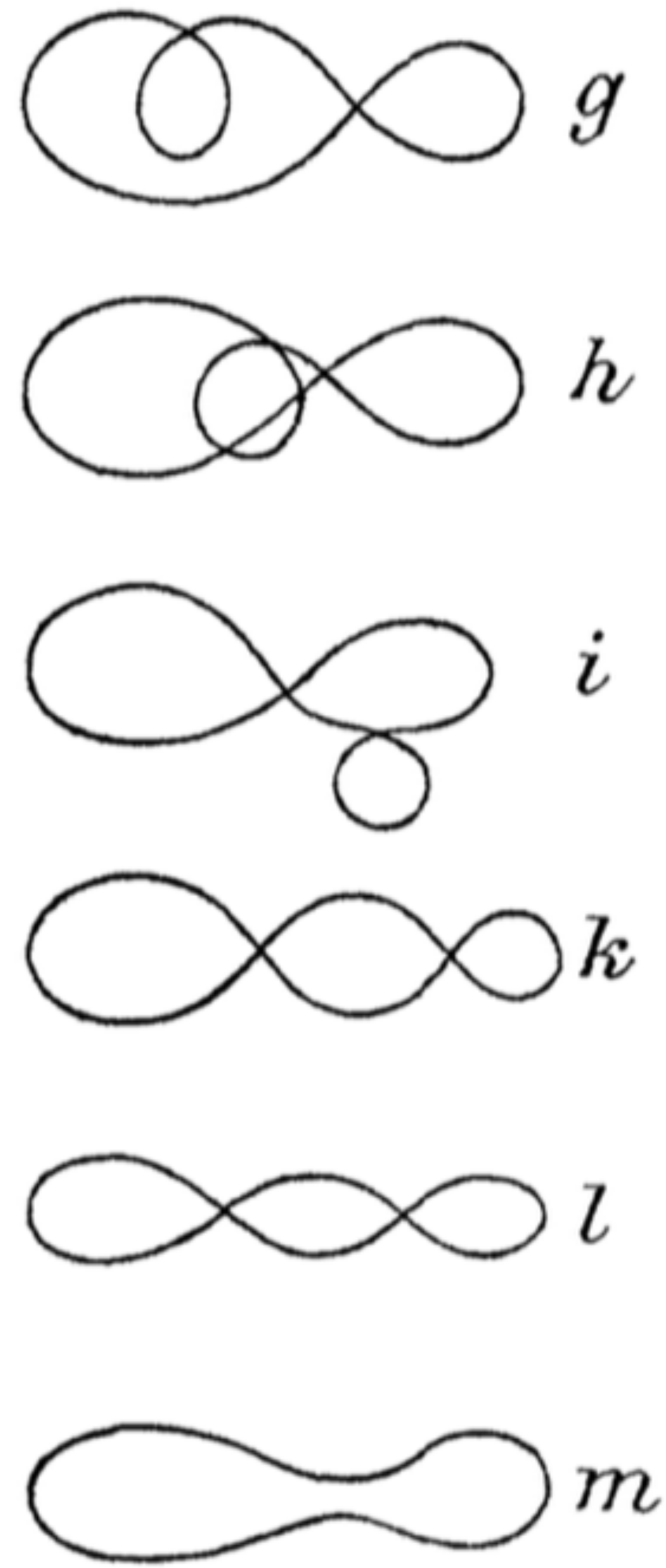
Boy (1933)

- In fact, positive and negative loops can ***literally*** cancel each other!



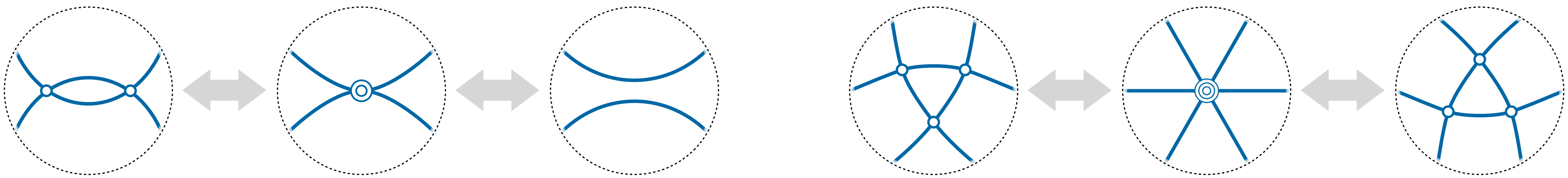
Whitney (1936)

- A ***regular*** curve has a unique non-zero tangent vector at every point.
- A ***regular homotopy*** is a continuous deformation through regular curves.
- **Whitney-Graustein Theorem:** Two curves are regularly homotopic if and only if they have the same rotation number.

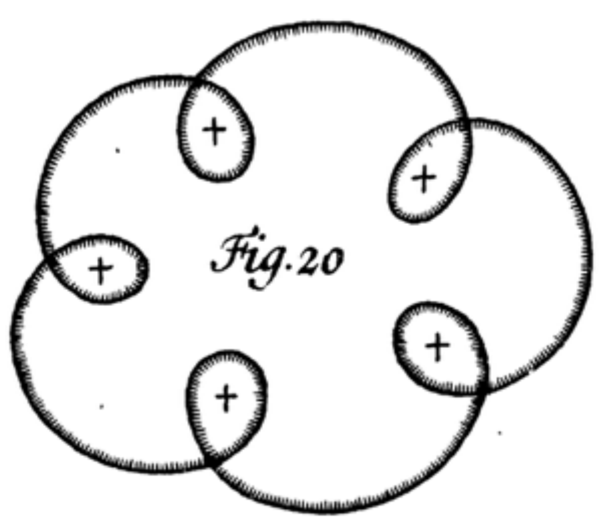


Modern proof of Whitney-Graustein [Francis 1971]

- Any regular homotopy can be decomposed into elementary moves. [Cf. Alexander and Briggs 1926, Reidemeister 1927]



- For any regular curve, there is a sequence of $O(n^2)$ moves that leads to a canonical curve with same rotation number.



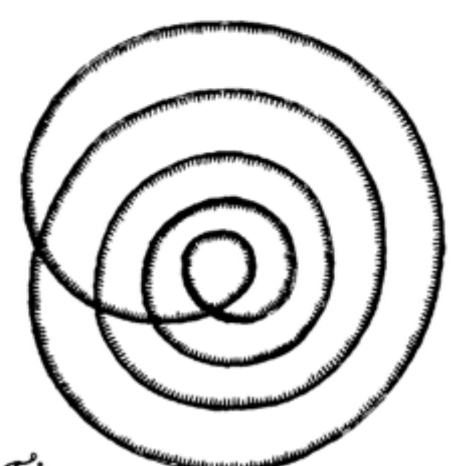
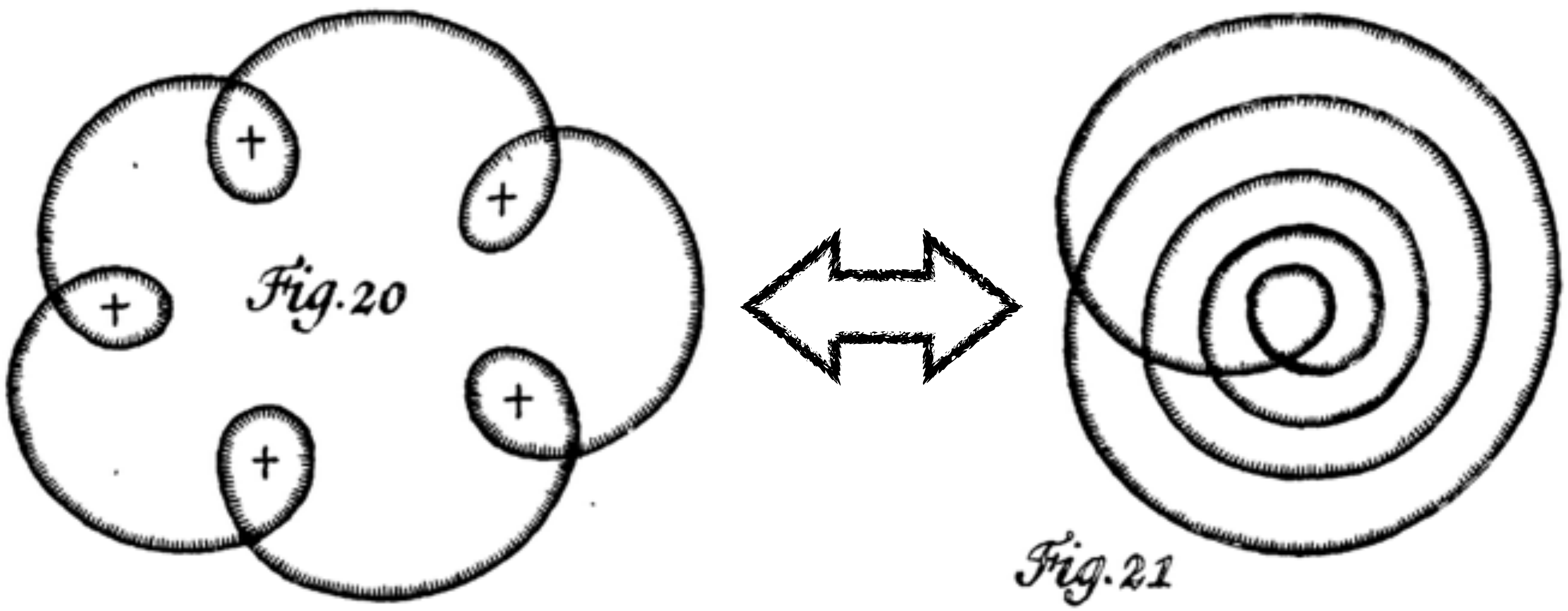


Fig. 21

Lower Bound

- In fact, $\Omega(n^2)$ moves are necessary in the worst case.
  [Arnold 1994, Nowik 2009]



Carl Friedrich Gauss (1777–1855)

Point in polygon algorithm

- Shoot a ray to the right. If the number of positive crossings equals the number of negative crossings, then the point is outside; otherwise, the point is inside.

Eine interessante Aufgabe scheint zu sein, die Bedingung analytisch anzugeben, ob ein gegebener Punkt innerhalb oder ausserhalb der Figur fällt.

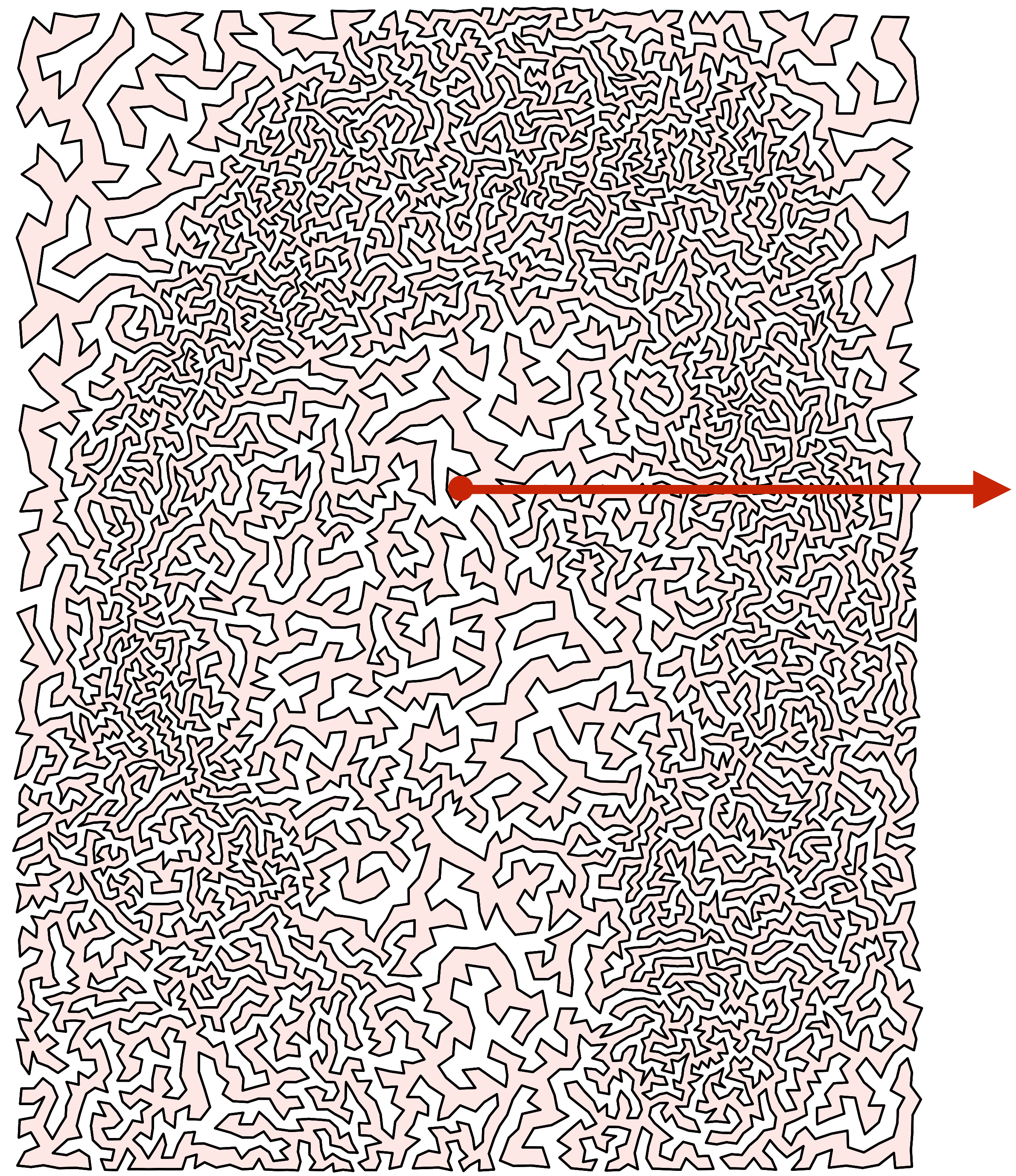
Die Auflösung ist leicht. Indem man den Punkt zum Anfangspunkt der Coordinaten wählt, zähle man alle Punkte

$$\begin{array}{llccc}
\alpha, & \text{wo} & y, & -y', & xy'-yx' \\
\beta, & \text{wo} & y, & -y', & yx'-xy' \\
\gamma, & \text{wo} & -y, & y', & xy'-yx' \\
\delta, & \text{wo} & -y, & y', & yx'-xy'
\end{array}$$

positiv sind; man hat dann

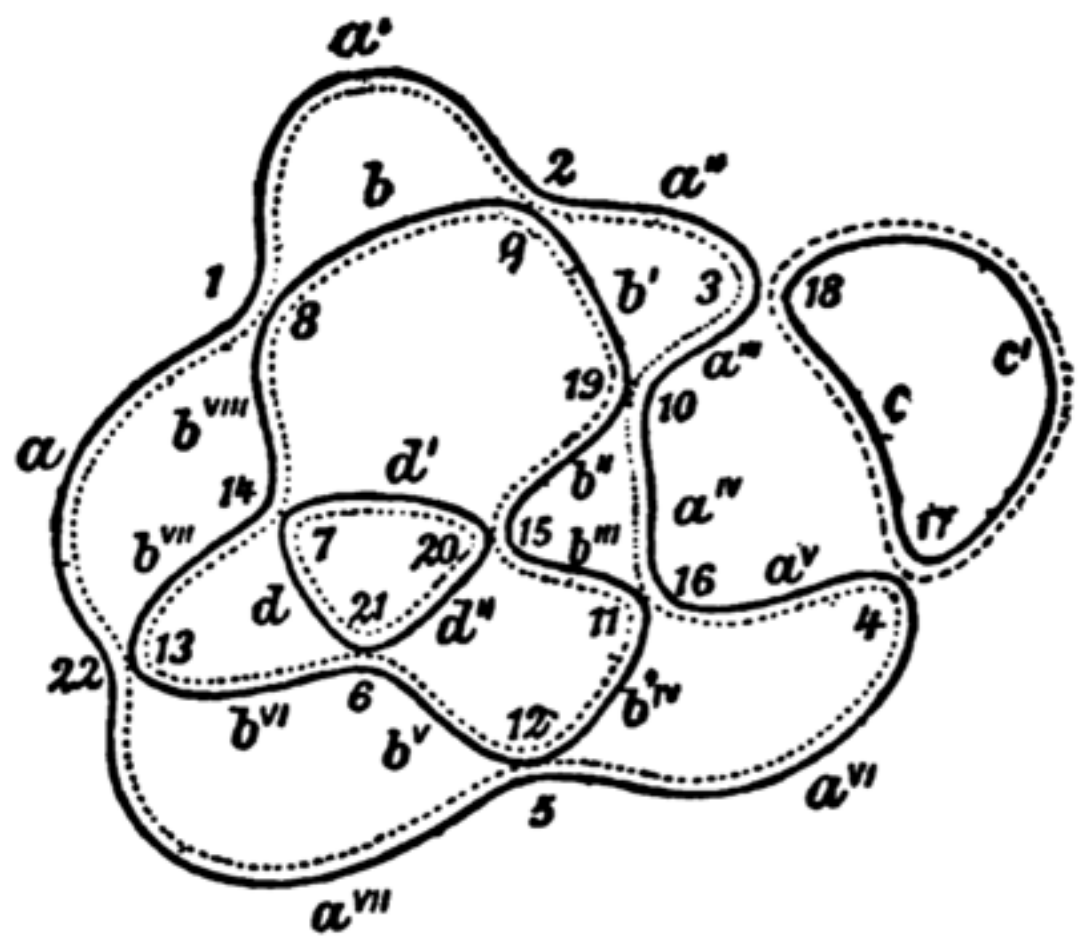
$$\alpha = \gamma, \qquad \beta = \delta.$$

Ist nun $\alpha - \beta = 0$, so liegt der Punkt ausserhalb, ist $\alpha - \beta = \pm 1$, so liegt er innerhalb.



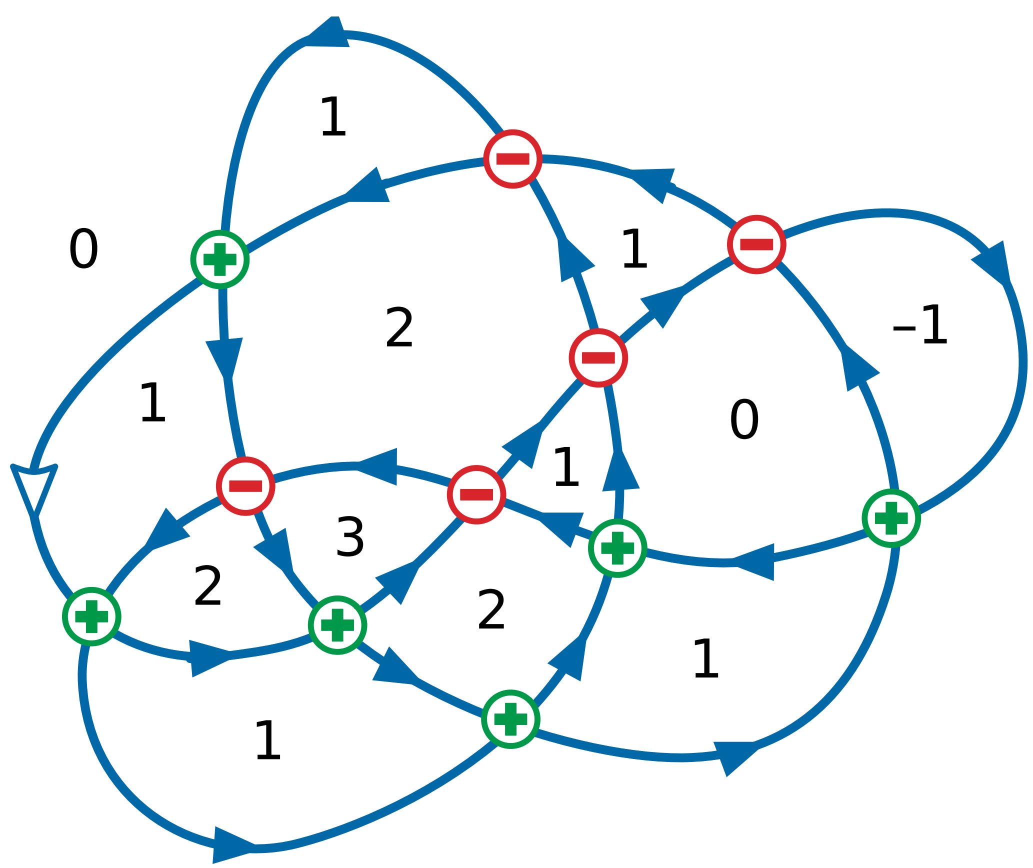
Seifert decomposition

- Uncross the curve at every vertex, preserving orientation
- Rotation number = sum of individual rotation numbers
- Winding number = sum of individual winding numbers



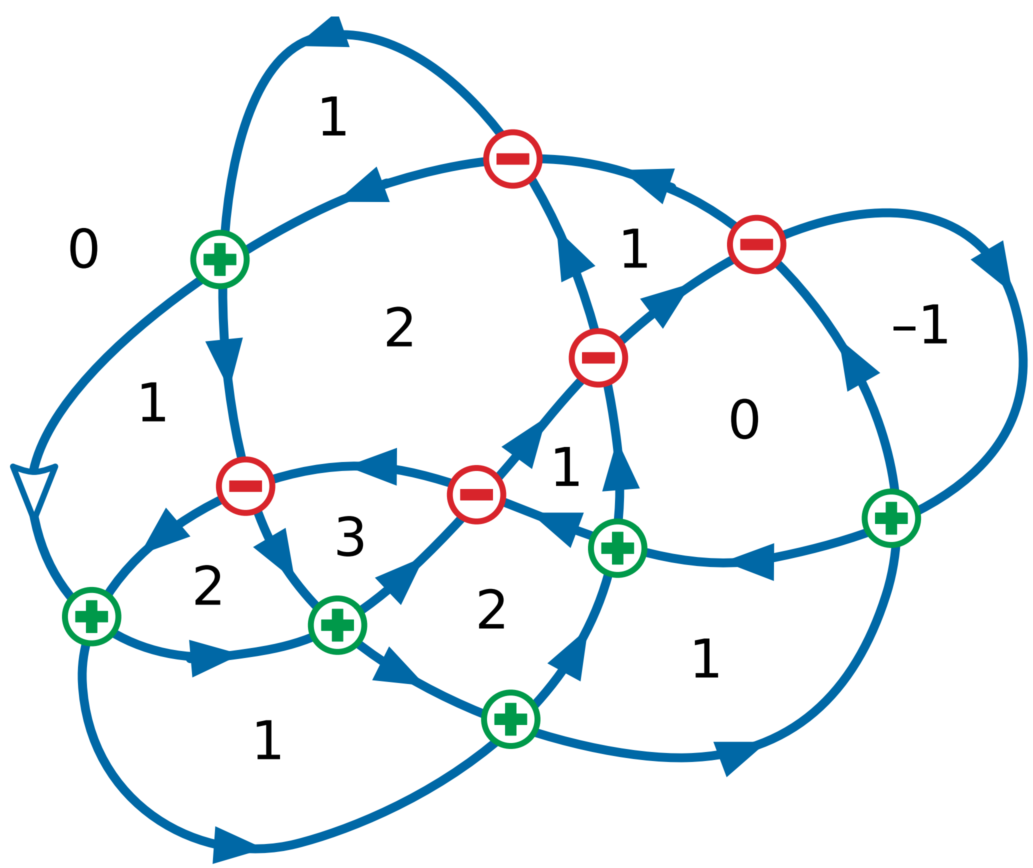
Signed vertices

- *positive* vertex = first crossing at that point is right to left (increasing winding number)
- *negative* vertex = first crossing at that point is left to right (increasing winding number)



Rotation number formula

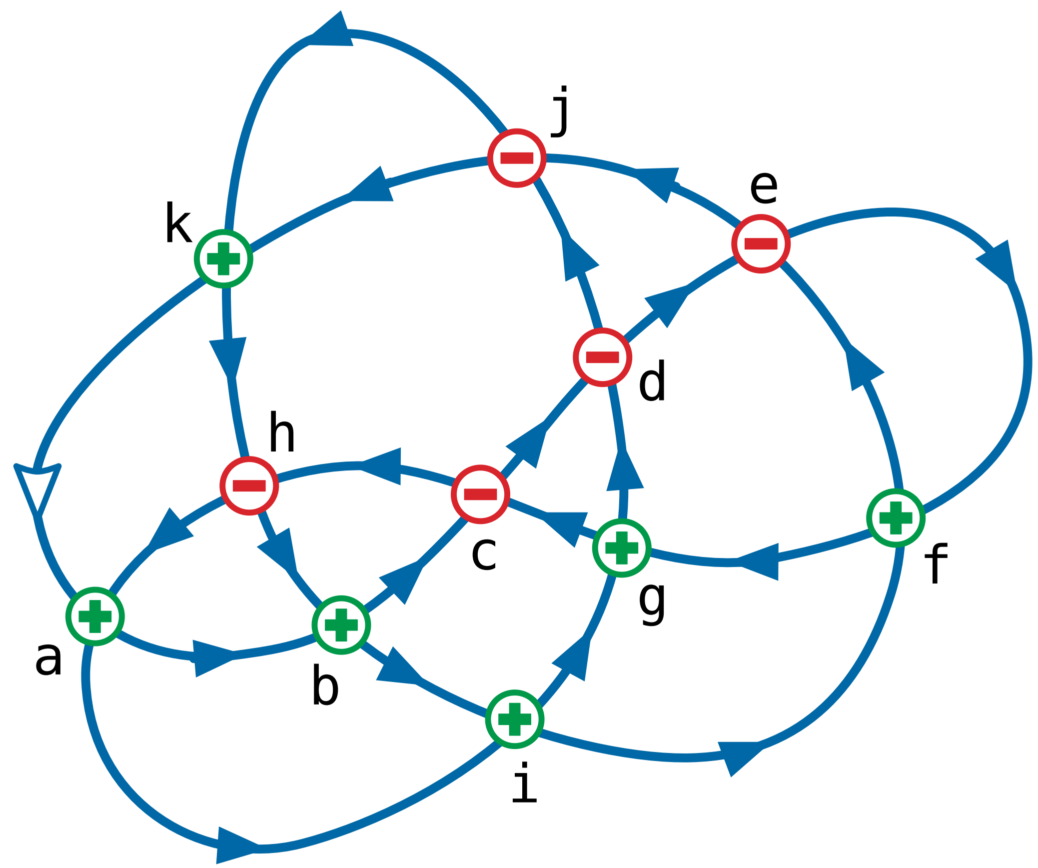
- Suppose basepoint is the leftmost point on the curve
- Then $\mathbf{rot}(C) = \sum_x \mathbf{sgn}(x) \pm \mathbf{1}$
  - +1 if tangent at basepoint is downward
  - –1 if tangent at basepoint is upward



Gauss word

- Sequence of crossing labels, either with or without signs

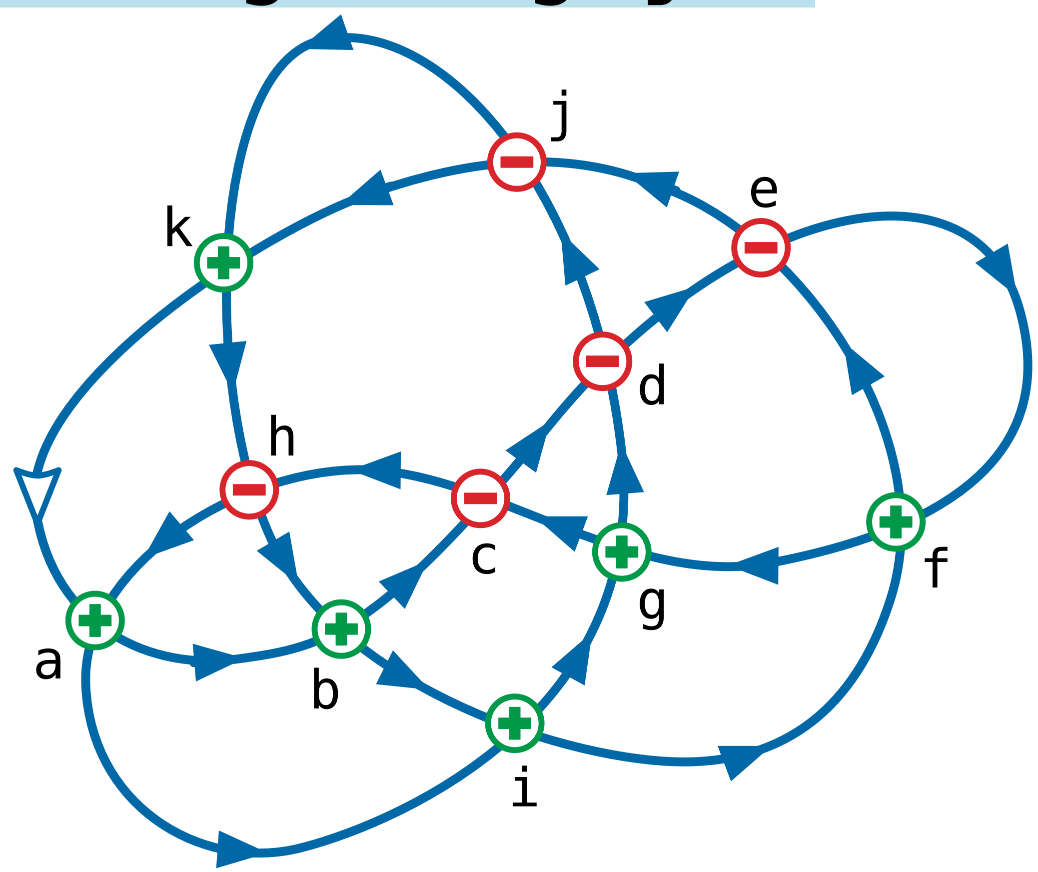
```
++---+++--+-+-++---++-
abcdefgchaigdjkhbifejk
```



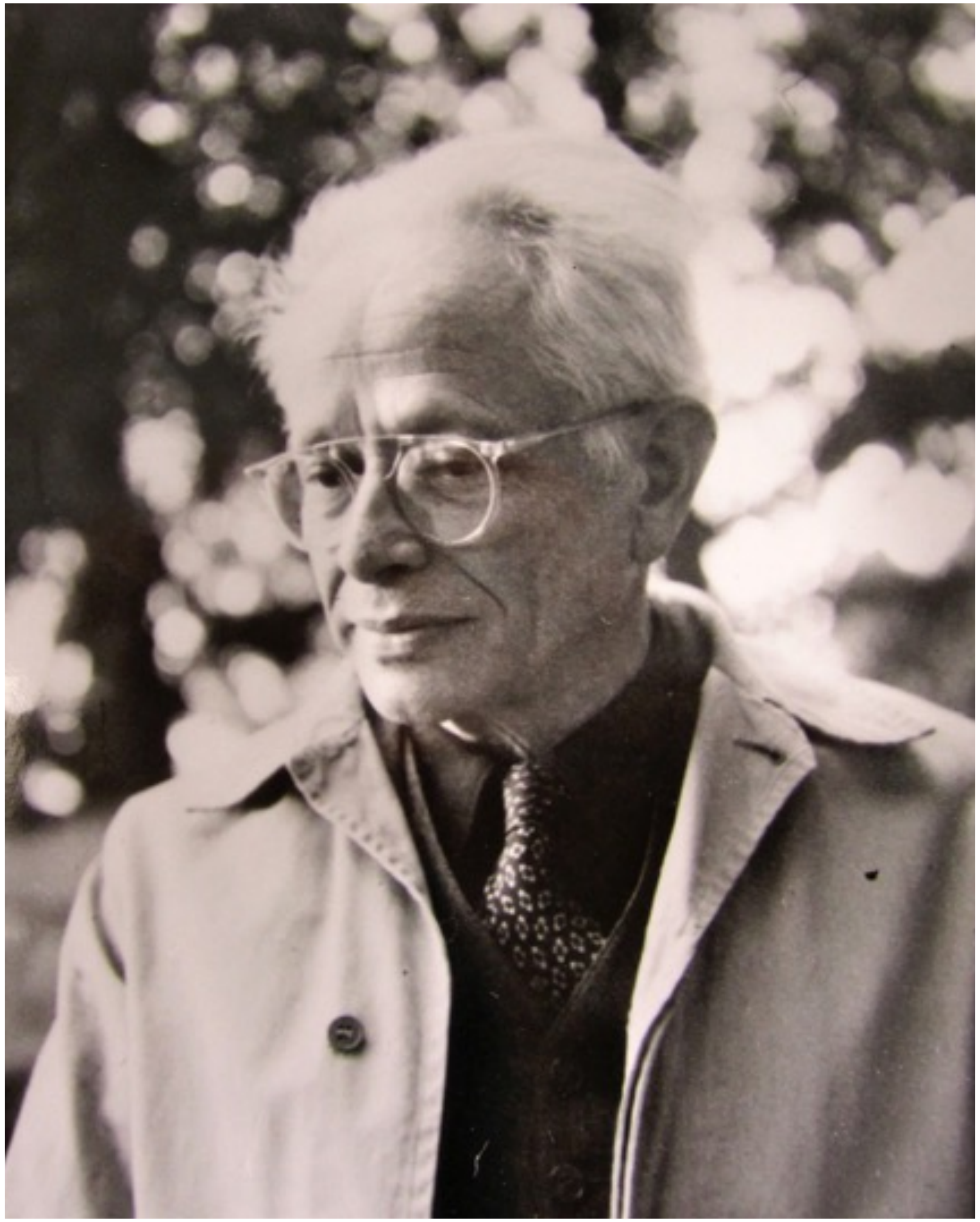
Gauss's problem

- When does a given Gauss code represent a planar curve?
- Necessary but not sufficient: Any pair of matching symbols separated by an even number of other symbols

`abcdefgchaigdjkhbifejk`

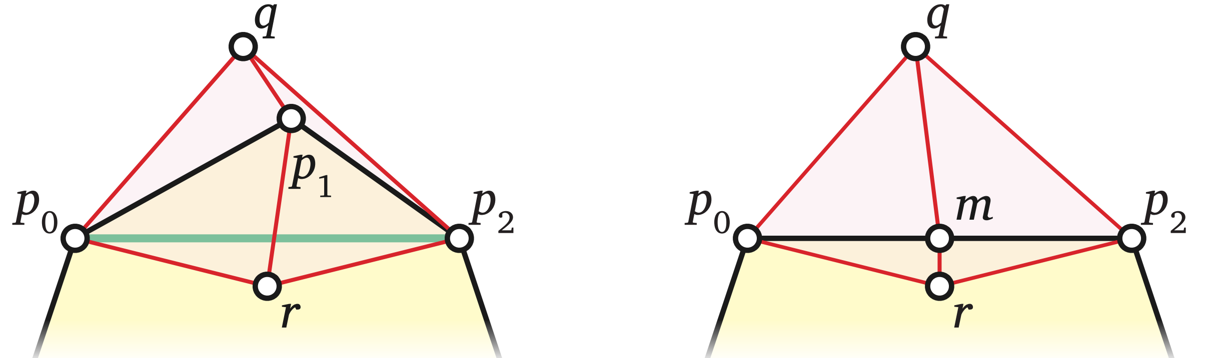


Max Dehn (1878–1952)



Dehn 1899

- First proof of the Jordan-Schönflies theorem for simple *polygons*: For any simple polygon P in the plane, there is a homeomorphism h:$\mathbf{R}^2 \to \mathbf{R}^2$ such that $h(P)$ is a triangle.
  - **Lemma:** Every polygon has a triangulation. (You know the proof!)
  - **Lemma:** Every polygon has an ear. (You know the proof!)
  - Constructs an appropriate map h by induction



Dehn and Heegaard 1907

III AB 3. ANALYSIS SITUS.

Von

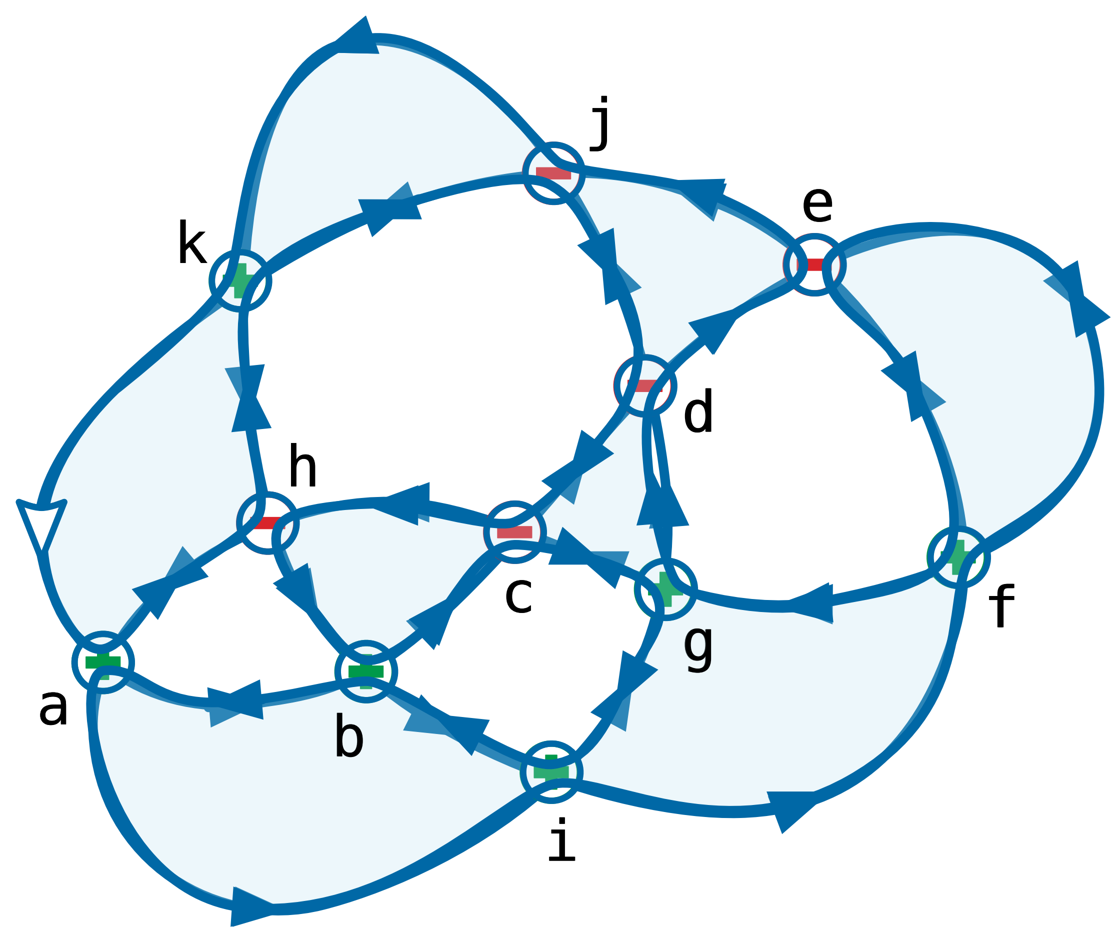
M. DEHN UND **P. HEEGAARD***)

IN MÜNSTER I/W. IN VEDBAEK B/KOPENHAGEN.

Dehn on Gauss' problem (1936)

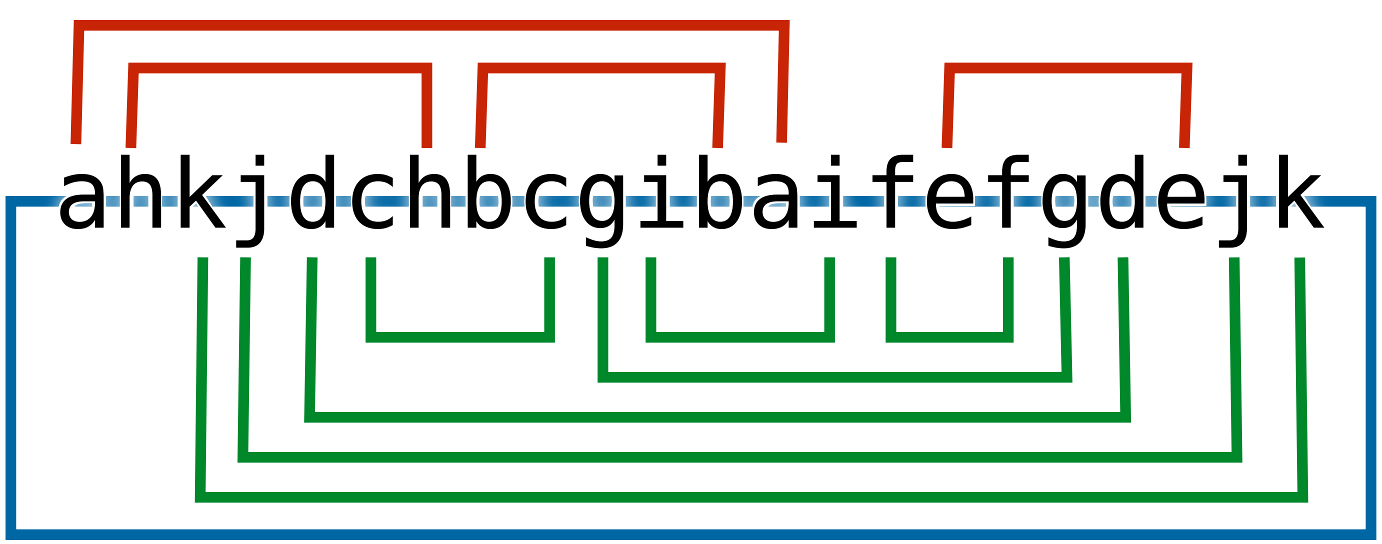
- *Untangle* the Gauss code by reversing each substring bounded by matching symbols

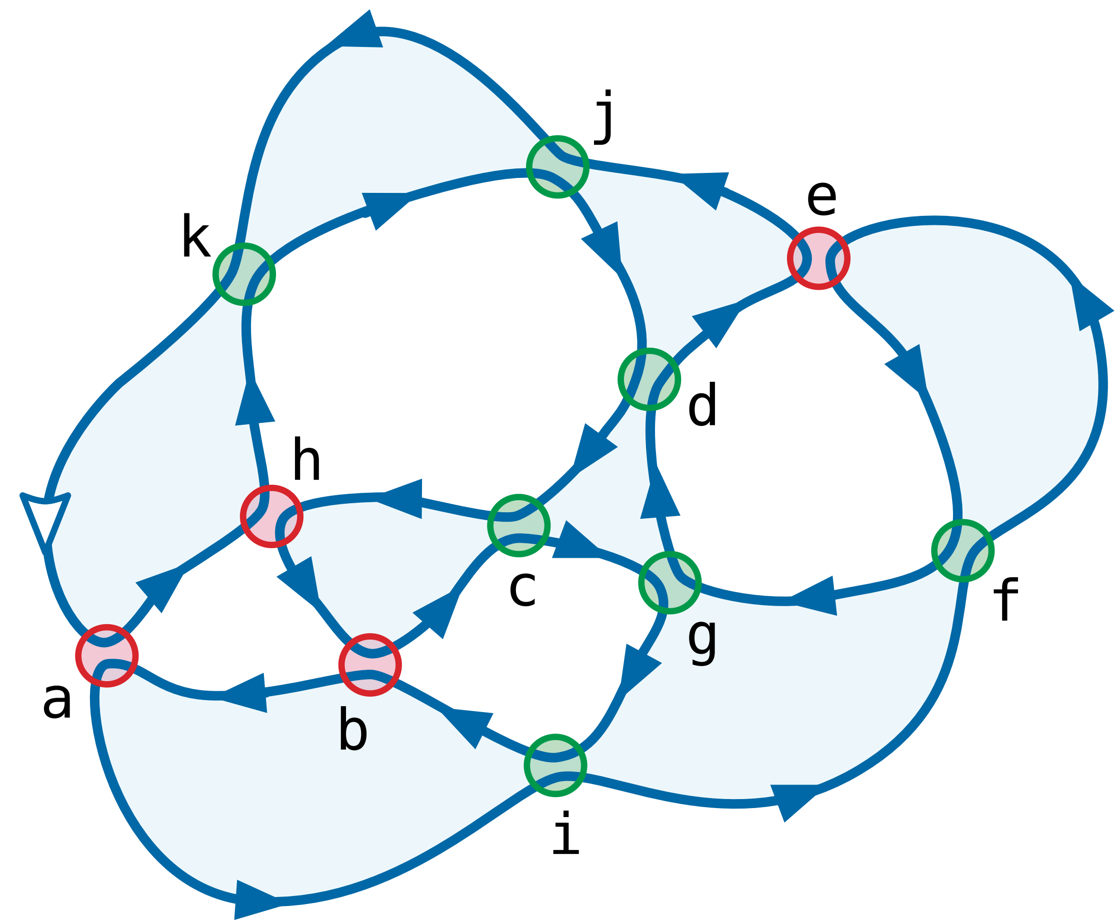
abcdefgchaigdjkhbifejk
ahcgfedcbaigdjkhbifejk
ahcgfedc**bhkjdgiab**ifejk
ah**cdefgc**bhkjdgiabifejk
ahc**djkhbcgfed**giabifejk
ahcdjkhbcgf**efibaigde**jk
ahcdjkhbcg**fef**ibaigdejk
ahcdjkhbc**giabifefg**dejk
a**hkjdch**bcgiabifefgdejk
ahkjdchbcg**ibai**fefgdejk
ahk**jedgfefiabigcbhcdj**k
ah**kjdchbcgibaifefgdejk**



Dehn on Gauss' problem (1936)

- Necessary but not sufficient: The *interlacement graph* of the untangled Gauss code is bipartite
- Equivalently: Untangled cycle + matching is a planar graph





Modern solution

- To make Dehn's condition necessary and sufficient, add duplicate symbols when untangling
  [de Fraysseix and Ossona de Mendez 1999]

abcdefgchaigdjkhbifejk

aAhcgfedcbAaigdjkhbifejk

aAhcgfedc**bBhkjdgiaABb**ifejk

aAh**cCdefgCc**bBhkjdgiaABbifejk

aAhcC**dDjkhBbcCgfeDd**giaABbifejk

aAhcCdDjkhBbcCgf**eEfibBAaigdDEe**jk

aAhcCdDjkhBbcCg**fFEeFf**ibBAaigdDEejk

aAhcCdDjkhBbcC**gGiaABbifFeEFfGg**dDEejk

aA**hHkjDdCcHh**BbcCgGiaABbifFeEFfGgdDEejk

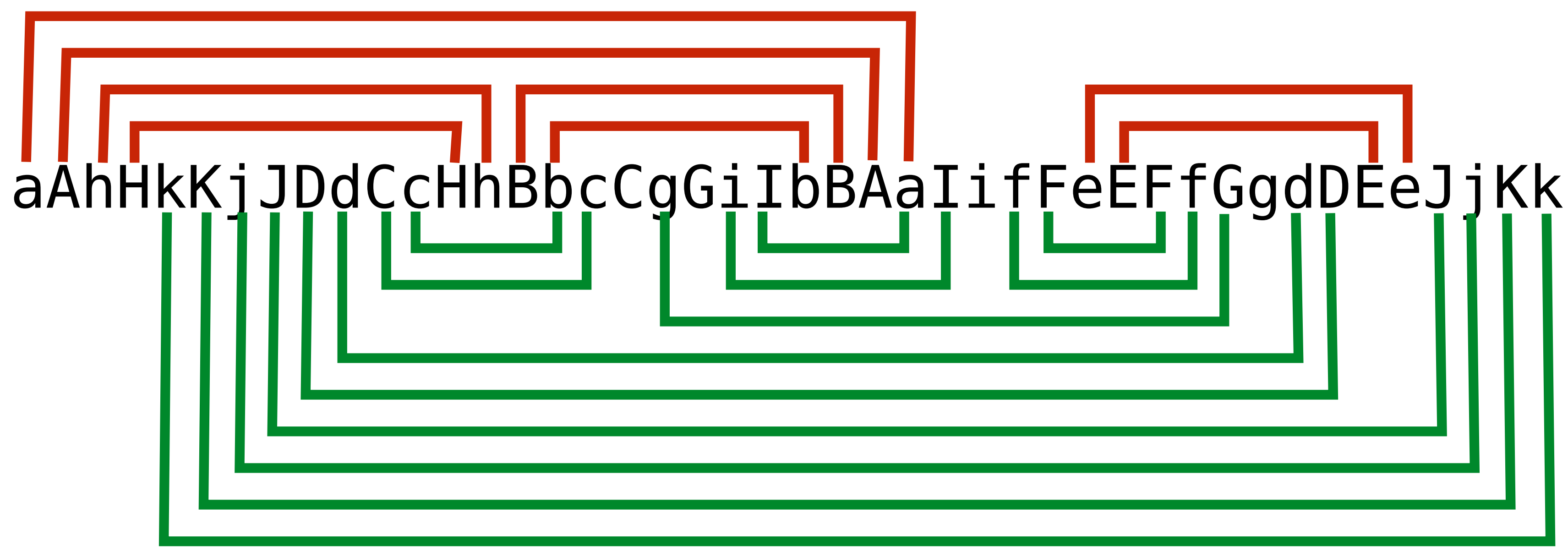
aAhHkjDdCcHhBbcCgG**iIbBAaIi**fFeEFfGgdDEejk

aAhHk**jJeEDdgGfFEeFfiIaABbIiGgCcbBhHcCdDJj**k

aAhH**kKjJDdCcHhBbcCgGiIbBAaIifFeEFfGgdDEeJjKk**

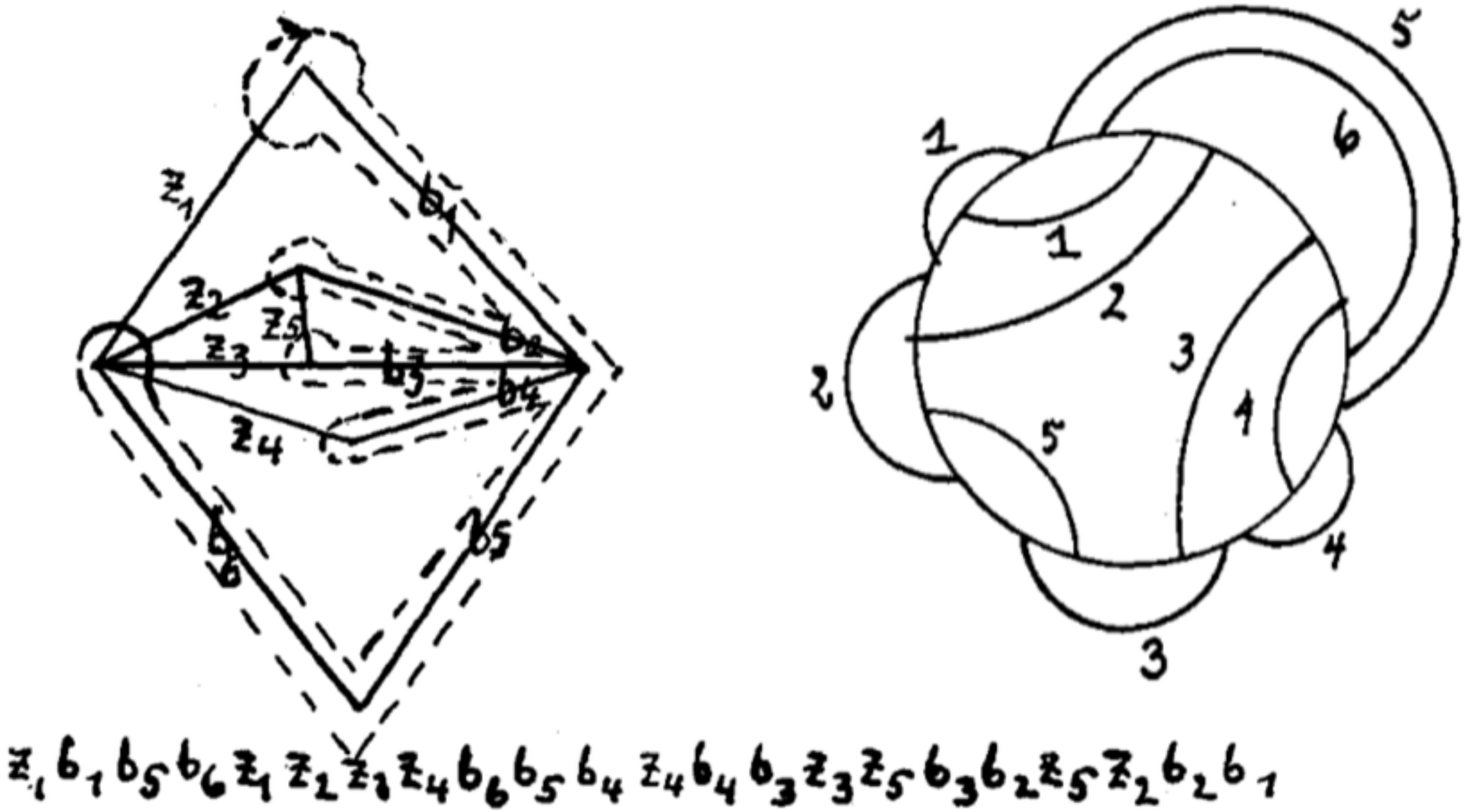
Modern solution

- To make Dehn's condition necessary and sufficient, add new symbols when untangling.
  [de Fraysseix and Ossona de Mendez 1999]



- The modified untangled Gauss code can be computed and tested for planarity in O(*n*) time.
  [Rosenstheil and Tarjan 1989]

Thank you!



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