Well-spaced samples of generic surfaces have sparse Delaunay triangulations

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Independent/joint work with
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Nice samples of nice surfaces have nice Delaunay triangulations

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Delaunay triangulations are neat!

Jeff E.

Surface reconstruction

Input: set P of *sample points* from an unknown smooth surface Σ

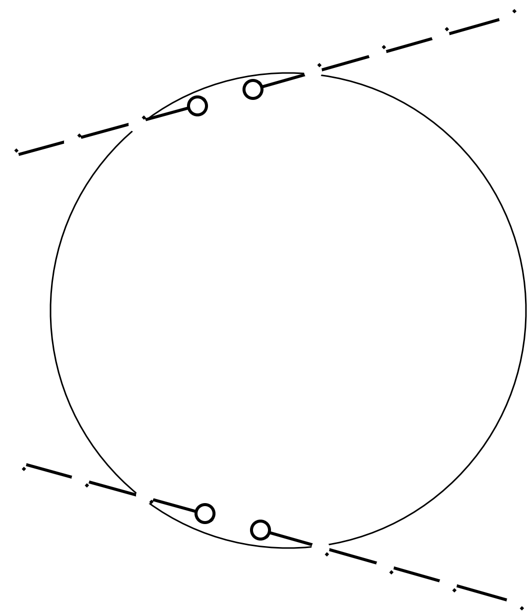
Output: an geometric approximation of Σ with the correct topology

Several provable algorithms
[Amenta, Bern, Boissonnat, Cazals, Dey, Edelsbrunner, Eppstein, Funke, Giesen, Hiyoshi, ...]

Lots of practical heuristics and improvements!

Delaunay triangulation

- ≤ 4 points form a *Delaunay simplex* if they lie on the boundary of an empty *Delaunay ball*
- n points in space can have $\Omega(n^2)$ Delaunay simplices



Theory ≠ Practice

Theory:
Delaunay triangulations have *quadratic* complexity (in the worst case).

Practice:
Delaunay triangulations have *linear* complexity.

Well, then it's not a very good "theory", is it?

Practical Delaunay complexity

Random points:

- in space: $O(n)$ [Meijering '53, Miles '72; Dwyer '91]
- on fixed convex polyhedron: $O(n)$
  [Golin and Na '00]
- on fixed nonconvex polyhedron: $O(n \log^4 n)$
  [Golin and Na '02]

Practical Delaunay complexity

(ϵ,k)-sample of Σ: Any ball of radius ϵ centered on Σ contains at least 1 and at most k samples

- fixed polyhedron: $O(k^2\ n)$
  [Attali and Boissonnat '01]⇐!!!
- arbitrary fixed surface: $\Theta(k^2\ n^{3/2})$
  Lower bound: [E'01], Upper bound: [E'02]
- **New:** fixed *generic* surface: $O(k^2\ n \log n)$

Warning: fixed surfaces

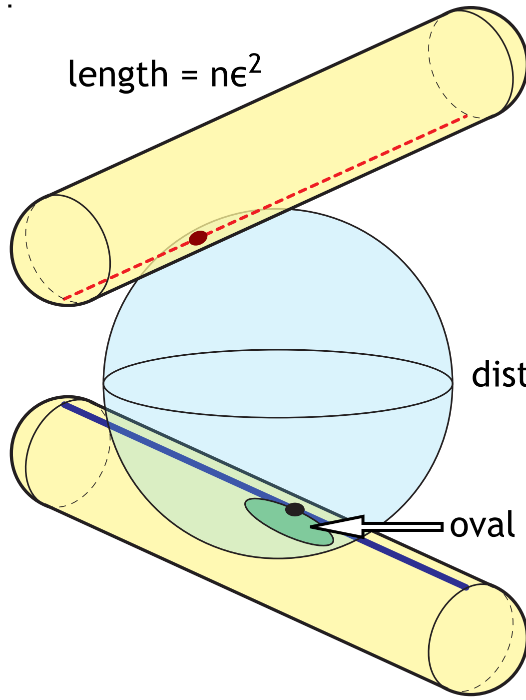
Any finite quantity that depends on the fixed surface Σ, but not on ϵ or k or any particular point, is considered a *constant*.

- surface area
- number of facets
- aspect ratios of facets
- angles between facets
- angles between edges
- diameter
- minimum local feature size
- min and max principal curvatures
- bounds on partial derivatives
- "genericity"

These constants are hidden in the O() notation.

"Delaunay condition number"

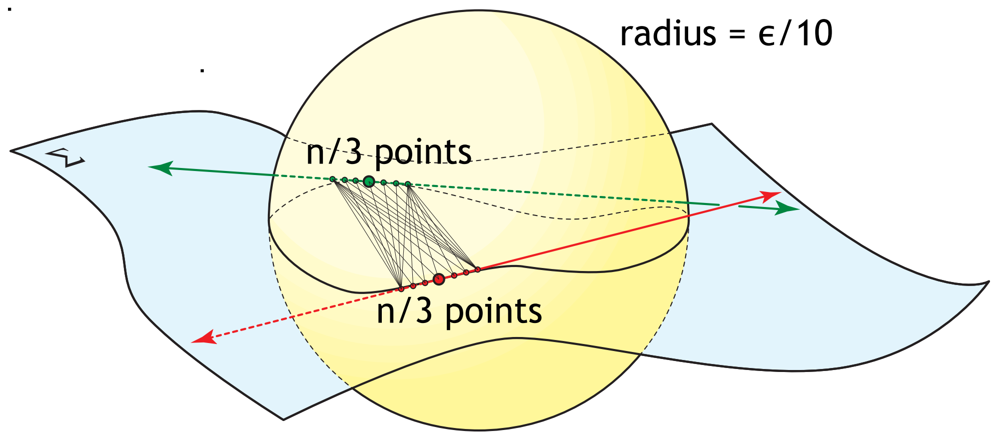
Why fix the surface?



If the surface varies with n and ϵ,
we can get $\Omega(n^2\epsilon^2)$ Delaunay simplices.

[E'01]

Why limit sample density?



For **any** surface except the sphere, we can get $\Omega(n^2)$ Delaunay simplices by locally oversampling.

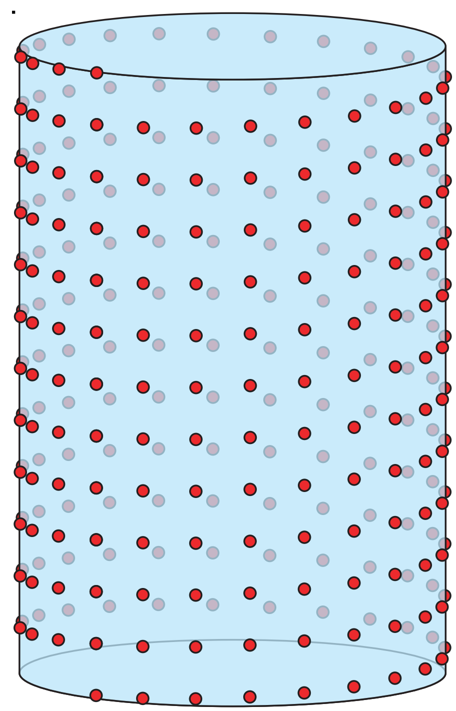
[E'01]

The helix

ϵ-sample of a cylinder:
$\sqrt{n}$ turns of a helix,
$\sqrt{n}$ points on each turn

Two points are Delaunay neighbors iff they are less than a full turn apart.

$\Omega(n^{3/2})$ Delaunay simplices!



[E'01]

Spread Δ

Spread = diameter/closest pair distance
[Goodman, Pollack, and Sturmfels '89; Valtr *et al*. '93-'97]

Roughly related to dimensionality:

- nicely distributed in a volume ⇔ $\Delta \approx n^{1/3}$
- nicely distributed on a surface ⇒ $\Delta \approx n^{1/2}$
- nicely distributed on a curve ⇒ $\Delta \approx n$

Spread upper bound

- The Delaunay triangulation of any set of points with spread Δ has complexity $O(\Delta^3)$.
- The Delaunay triangulation of the union of k sets, each with spread Δ, has complexity $O(k^2\Delta^3)$.
- An (ϵ,k)-sample of a fixed surface is the union of k sets with spread $\Theta(\sqrt{n})$, so its Delaunay complexity is $O(k^2n^{3/2})$.

[E'02]

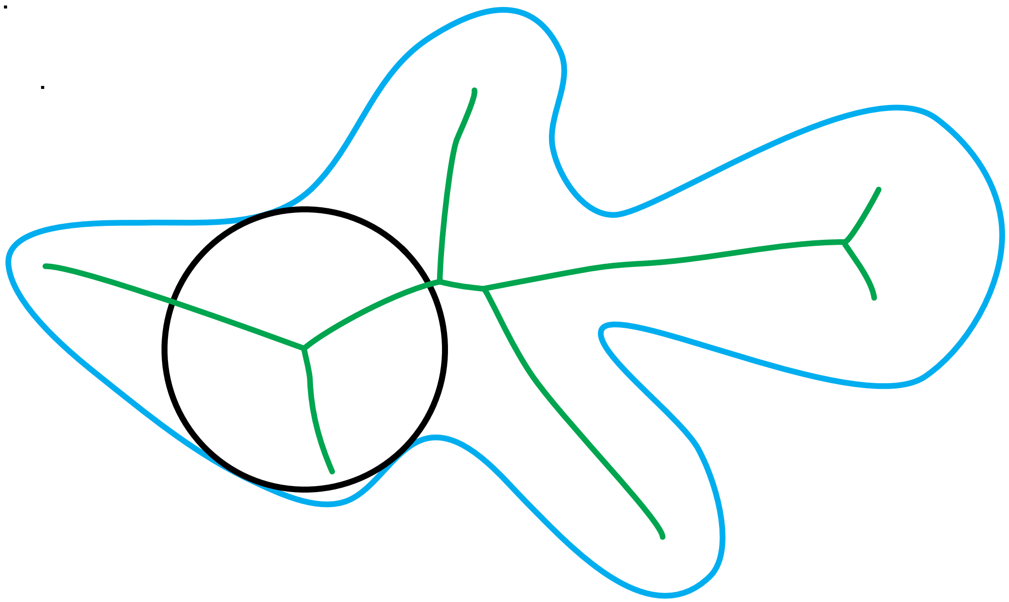
Theory ≠ Practice

Theory:
Delaunay triangulations of nice surface samples have complexity $\Theta(n^{3/2})$ in the worst case, and the worst case example is simple!

Practice:
Okay, sure, but Delaunay triangulations of *real* surface samples always have *linear* complexity!

So it's still not a very good "theory", is it?

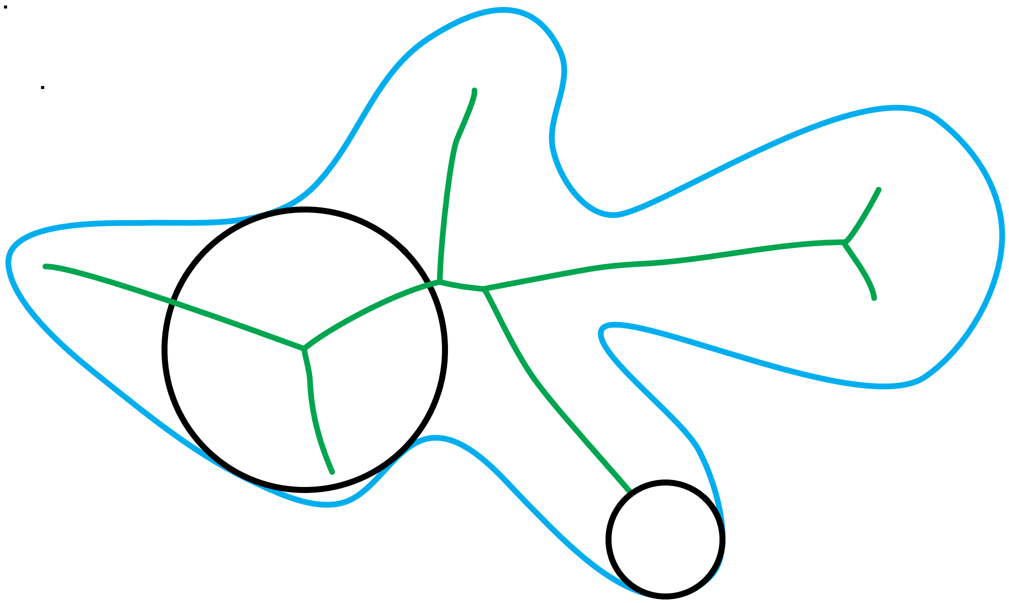
Medial axis



medial ball: empty interior, touches Σ more than once

medial axis: centers of medial balls

Generic contact types



A_1: simple tangency

A_3: osculation ⇒ local maximum of curvature

Generic surfaces

- Σ is *generic* if no medial ball touches Σ more than four times, counting with multiplicity
- Generically, only A_1 and A_3 contacts.
- A_3 contacts lie on *ridge curves* on Σ: local max of principal curvature
- Almost every surface is generic, but *not* surfaces of revolution or Herbert’s skin.

Generic medial axes

- Exactly 5 generic medial axis features
  [Bryzgalova ‘77; Mazov ‘82; Bruce, Giblin, and Gibson ‘85; Bogaevsky ’89; Giblin and Kimia ‘00; Leymarie and Kimia ‘03]
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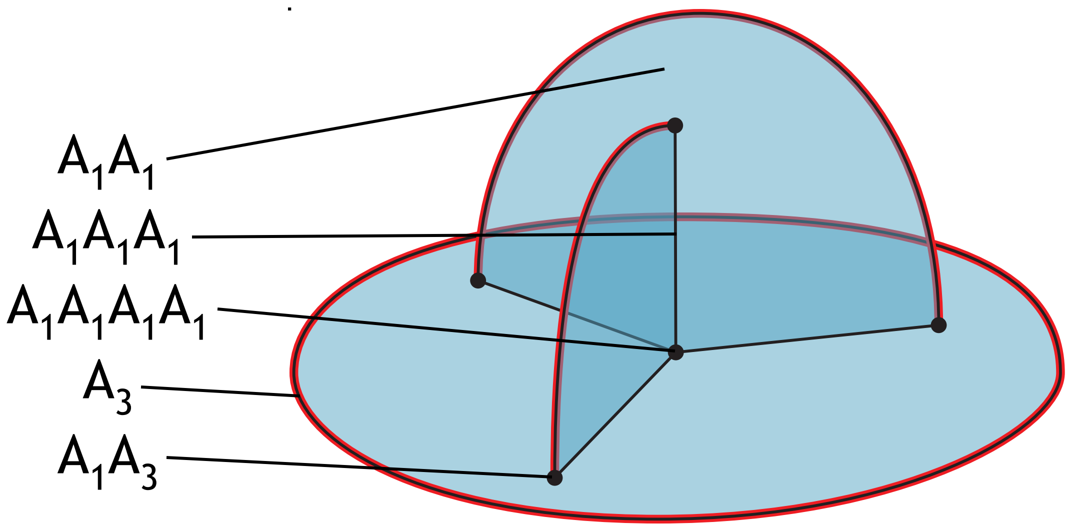


Intuition

As the sampling density increases, Delaunay balls behave more and more like medial balls.

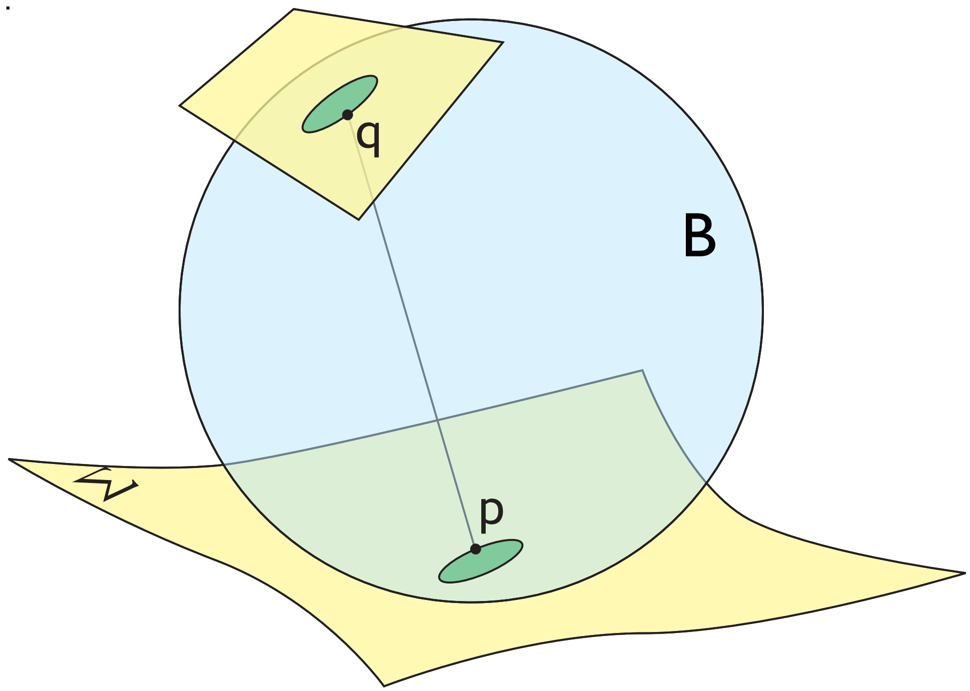
So to understand the *combinatorial* behavior of Delaunay balls in the limit as $\epsilon \to 0$, we need to study the *differential* behavior of medial balls!

Curvature measures

- $r(p)$ = radius of medial ball tangent at p
- $\kappa_1(p)$ = maximum principal curvature at p
- $\kappa_2(p)$ = minimum principal curvature at p

- $\kappa_2 < \kappa_1 < 1/r$ except
  - $\kappa_2 = \kappa_1 < 1/r$ only at umbilic points
  - $\kappa_2 < \kappa_1 = 1/r$ only at A_3 contact curves

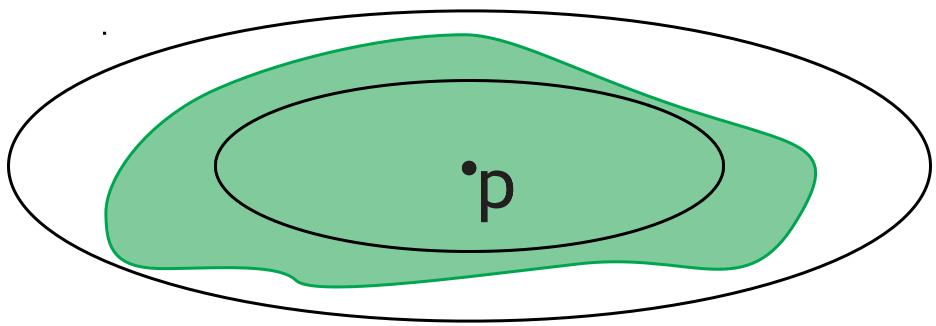
Delaunay ball intersecting surface



Expand a medial ball tangent at some point p far from A_3 contact curves, so $\kappa_1 < 1/r$.

Taylor series approximation ⇒ $B \cap \Sigma$ fits between two ellipses with aspect ratio

$$\sqrt{\frac{1 - r\kappa_2}{1 - r\kappa_1}}$$

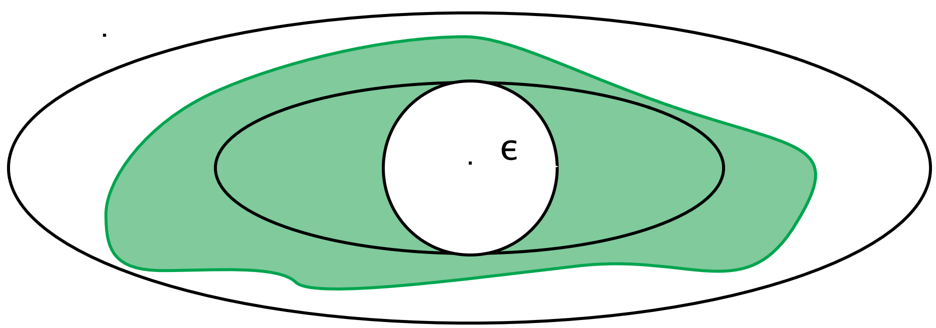


B is a Delaunay ball

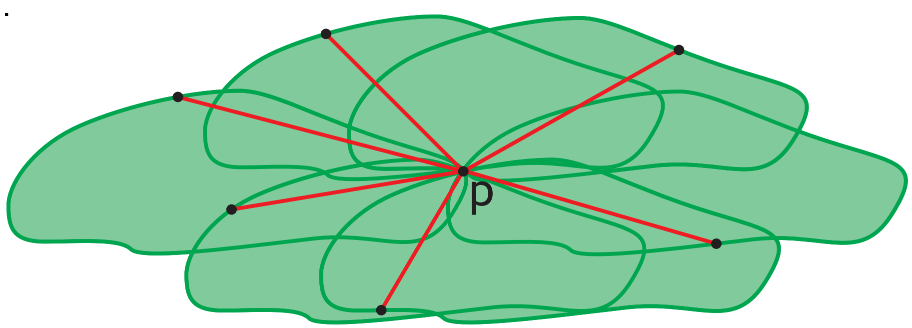
⇒ small ellipse can't contain ϵ-disk

⇒ larger ellipse can't contain 2ϵ-disk

Area of blob < Area of larger ellipse < $4\pi\varepsilon^2\sqrt{\frac{1 - r\kappa_2}{1 - r\kappa_1}}$



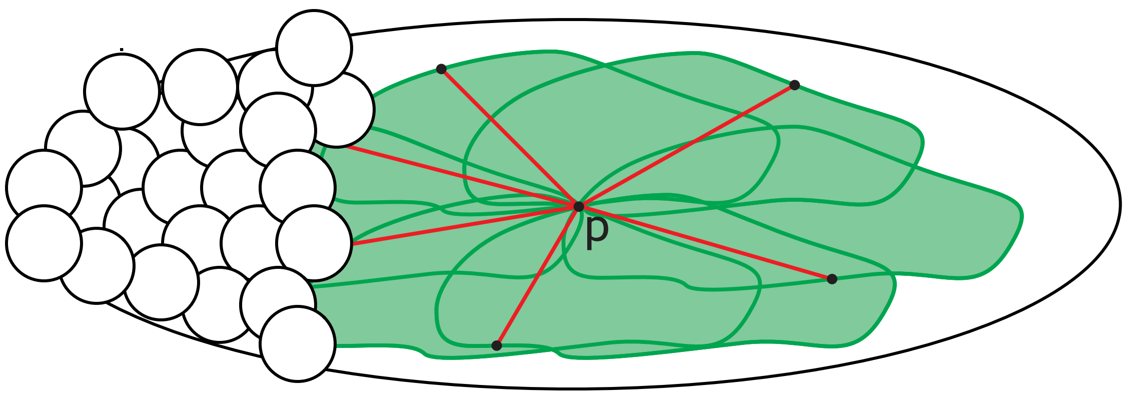
- Let p be a sample far from A_3 contact curves.
- Every local Delaunay neighbor of p carves out an empty blob on Σ.
- Since local neighbors are close, these blobs all look about the same.



- Union of all blobs fits in ellipse with area

$$64\pi\varepsilon^2\sqrt{\frac{1 - r\kappa_2}{1 - r\kappa_1}} = O(\varepsilon^2)$$

- Large ellipse covered by O(1) ϵ-balls, each containing O(1) sample points



Lemma:

Any point far from A_3 contact curves has O(1) local Delaunay neighbors.

Near A_3 contact curves

Suppose p is distance x from an A_3 curve

- $x > \sqrt{\epsilon}$:
  $1-\kappa_1 r = \Theta(x^2) \Rightarrow O(1/x)$ local neighbors
- $x \leq \sqrt{\epsilon}$
  higher-order Taylor approximation $\Rightarrow$
  $O(1/\sqrt{\epsilon})=O(n^{1/4})$ local neighbors

Now integrate over x...

Danger!

Theorem:

O(n log n) local Delaunay edges

However, points near A_3 curves might still have high degree!

What's left to do?

- Count *remote* Delaunay edges, which cross from one side of the surface to the other
  - Remote neighborhood is only O(1) bigger than local neighborhood.
- Count *external* Delaunay edges, whose Delaunay balls are centered outside Σ.
  - Apply a conformal transformation to turn the surface inside out!

Main Result

Fix a generic surface Σ.

The Delaunay triangulation of any (ϵ,O(1))-sample of Σ has complexity O(n log n).

Future work

What is the expected complexity of the Delaunay triangulation of a set of *random* points from a surface?

- Theorem: For cylinder, $\Theta(n \log n)$!
- Conjecture: For any generic surface, $\Theta(n)$
- Conjecture: For any ~~smooth~~ surface, ~~$\Theta(n \log n)$~~
  $\alpha\, n \log n + \Theta(n)$ for some small absolute constant α (independent of Σ)

Thanks for listening, and thanks to the organizers for a great workshop!