Untangling Planar Curves and Planar Graphs

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There's a certain type of brain that's easily disabled.

If you show it an interesting problem, it involuntarily drops everything else to work on it.

This has led me to invent a new sport: nerd sniping.

See that physicist crossing the road?

Hey!

On this infinite grid of ideal one-ohm resistors, what's the equivalent resistance between the two marked nodes?

It's... Hmmm. Interesting. Maybe if you start with... no, wait. Hmmm... you could—

Fooooom

I will have no part in this. C'mon, make a sign. It's fun! Physicists are two points, mathematicians three.
Homology
Homology
Der Man mit dem Mundwerk, Paul Klee (1930)
Homotopy moves

1 → 0
2 → 0
3 → 3

Diagram:
- Starting point
- First move
- Second move
- Third move
Homotopy moves

1→0

2→0

3→3

How many?
Previous bounds

- \(O(n^2)\) homotopy moves are always sufficient
  
  [Steinitz 1916; Grünbaum 1967; Francis 1971; Feo 1985; Truemper 1989; Vegter 1989; Feo Provan 1993; Hass and Scott 1994; Nowik 2000; ...]
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  [Steinitz 1916; Grünbaum 1967; Francis 1971; Feo 1985; Truemper 1989; Vegter 1989; Feo Provan 1993; Hass and Scott 1994; Nowik 2000; …]

- $\Omega(n)$ homotopy moves are sometimes necessary
  
  [trivial]
$\Theta(n^{3/2})$
Electrical transformations

- Loop reduction
- Parallel reduction
- Δ→Y transformation
- Leaf reduction
- Series reduction
- Y→Δ transformation
Degree-1 reduction  Series-parallel reduction  ΔY transformation
Resistor network analysis [Kennelly 1899]
Shortest paths [Akers 1960]

\[ R = \min\{r, s+t\} \]
\[ S = \min\{s, r+t\} \]
\[ T = \min\{t, r+s\} \]
Maximum flows [Akers 1960]

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Resistor network analysis [Kennelly 1899]

AC circuit analysis [Russell 1904]

Shortest paths and maximum flows [Akers 60]

Network reliability estimation
[Lehman 63, Traldi 83, Chari Feo Provan 96, Truemper 02]

Multicommodity flows [Feo 85]

Counting spanning trees, perfect matchings, and cuts
[Colbourn Provan Vertigan 95]

Generalized Laplacian linear systems
[Gremban 96, Nakahara Takahashi 96]

Circular planar networks
[Colin de Verdière, Gitler, Vertigan 96; Curtis, Ingerman, Morrow 98]

Kinematic analysis of articulated robots [Staffelli Thomas 02]

Flow estimation from noisy measurements [Zohar Geiger 07]
Any *planar* graph can be reduced using a *finite number* of electrical transformations. [Steinitz 1916, Epifanov 1966]

How many?
Previous bounds

- $O(n^2)$ electrical transformations are always sufficient
  [Steinitz 1916; Epifanov 1966; Grünbaum 1967; Feo 1985; Truemper 1989; Gitler 1991; Feo Provan 1993; Colin de Verdière, Gitler, Vertigan 1996; Song 2001; ...]

- $\Omega(n)$ electrical transformations are always necessary
  [trivial]
\( \Theta(n^{3/2}) \)
A graph is the 1-skeleton of a convex polytope in $\mathbb{R}^3$ if and only if it is planar and 3-connected.
Steinitz’s Theorem

A graph is the 1-skeleton of a convex polytope in \( \mathbb{R}^3 \) if and only if it is planar and 3-connected.

[Steinitz 1916]
Medial graph $G^*$

- Medial vertex for each edge of $G$
- Medial edge for each corner of $G$

[Tait 1877, Steinitz 1916]
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[Tait 1877, Steinitz 1916]
Medial electrical moves

[Steinitz 1916]
Medial electrical moves

[Steinitz 1916]
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Medial electrical moves

[Steinitz 1916]

Hey, these look familiar.
Steinitz’s Lemma

Every 4-regular plane graph contains either an *empty loop* or a *minimal bigon*.

[Steinitz 1916]
Steinitz’s Lemma

Every 4-regular plane graph contains either an empty loop or a minimal bigon.

[Steinitz 1916]
Unter den mit Randkanten behafteten Flächen einer irreduziblen Spindel kommen wenigstens zwei Dreiecke vor.
Every non-empty minimal bigon contains \textbf{at least two triangular faces} (one adjacent to each side).
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So we can reduce any minimum bigon to an empty bigon using $3 \rightarrow 3$ moves.
Steinitz’s Algorithm

- While there are vertices
  - If there is an empty loop, remove it with a $1 \rightarrow 0$ move
  - Otherwise, empty any minimal bigon with $3 \rightarrow 3$ moves, and then remove it with a $2 \rightarrow 0$ or $2 \rightarrow 1$ move

[Steinitz 1916]
Steinitz’s Algorithm

- While there are vertices
  - If there is an empty loop, remove it with a $1\to0$ move
  - Otherwise, empty any minimal bigon with $3\to3$ moves, and then remove it with a $2\to0$ or $2\to1$ move

- $O(n)$ moves per bigon = $O(n^2)$ moves

[Steinitz 1916]
Positive $3 \rightarrow 3$ moves

If there are no empty loops or empty bigons, then some $3 \rightarrow 3$ move decreases the sum of face depths.
Feo and Provan’s Algorithm

- While there are vertices
  - If there is an empty loop, perform a 1→0 move
  - Else if there is any empty bigon, perform a 2→0 or 2→1 move
  - Else perform any positive 3→3 move

- \( O(\Phi) = O(n^2) \) moves,
  where potential \( \Phi = \) sum of face depths
Homotopy Algorithm
Any loop can be removed using at most $3A$ homotopy moves, where $A$ is the number of interior faces.
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Contracting a loop decreases potential by at least $A$. 
New Algorithm

- While there are vertices, shrink any loop.
- $O(\Phi) = O(n^2)$ moves, where $\Phi = \text{potential}$
- **Tangle** = intersection of curve with a generic closed disk
- Boundary-to-boundary paths called *strands*
- Face depths and potential defined exactly as for curves.
Any tangle can be tightened in $O(md + ms)$ moves, where $m = \#\text{vertices}$, $d = \text{max face depth}$, and $s = \#\text{strands}$.
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- Remove all loops in $O(\Phi) = O(md)$ moves.
- Straighten strands in $O(ms)$ moves.
Any tangle can be *tightened* in $O(md + ms)$ moves, where $m = \#$vertices, $d = \text{max face depth}$, and $s = \#\text{strands}$

- Remove all loops in $O(\Phi) = O(md)$ moves.
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We call a tangle *useful* if \( m \geq s^2 \) and \( d = O(m^{1/2}) \)

- Can be tightened using \( O(md+ms) = O(m^{3/2}) \) moves
- Tightening removes at least half of the vertices
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▷ Tightening removes at least half of the vertices

**Lemma:** Every curve admits a useful tangle.

▷ Depth contours define a sequence of nested tangles

▷ Suppose \( i \)th tangle has \( m_i \) vertices and \( s_i \) strands

▷ If the first \( i \) tangles are all useless, then \( s_i \geq i/2 \) and thus \( m_i \geq i^2/4 \)
Our homotopy algorithm

- While there are vertices, tighten any *useful* tangle

**Analysis:**

- Tightening a useful tangle with $m$ vertices takes $O(m^{3/2})$ moves.
- Charge $O(m^{1/2})=O(n^{1/2})$ moves to each deleted vertex
- So removing all $n$ vertices takes $O(n^{3/2})$ moves.
Homotopy Lower Bound
**Defect** = unique curve invariant that is zero for simple curves and changes as follows under homotopy moves:

<table>
<thead>
<tr>
<th></th>
<th>1→0</th>
<th>2→0</th>
<th>3→3</th>
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<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>+2</td>
</tr>
</tbody>
</table>

Simplifying any curve \( \gamma \) requires at least \( |\text{defect}(\gamma)/2| \) homotopy moves.

[Aicardi 94, Arnold 94]
\[ \text{defect} = -2 \sum_{x \interleaved y} \text{sgn}(x) \text{sgn}(y) \]

- \(x \interleaved y\) means vertices \(x\) and \(y\) are \textit{interleaved}

- \(\text{sgn}(x) = +1\) if first pass through \(x\) crosses second from left to right; 
  \(\text{sgn}(x) = -1\) otherwise. \([\text{Gauss c.1830}]\)

- Independent of basepoint and orientation.

\[[\text{Polyak 98}]\]
Flat torus knot $T(p,q)$

$$(\cos(q\theta)+2) \cos(p\theta), (\cos(q\theta)+2) \sin(p\theta))$$

$T(7,8)$

$T(8,7)$
Flat torus knot $T(p,q)$

$$(\cos(q\theta)+2) \cos(p\theta), (\cos(q\theta)+2) \sin(p\theta))$$

$T(7,8)$

$T(8,7)$

defect($T(p, p+1)$) = $2\binom{p+1}{3}$

$= \Theta(n^{3/2})$

[Even-Zohar et al. 2014]
Flat torus knot $T(p,q)$

\[(\cos(q\theta)+2) \cos(p\theta), (\cos(q\theta)+2) \sin(p\theta))\]

$T(7,8)$

$T(8,7)$

$\text{defect}(T(p, p+1)) = 2 \left( p + 1 \right) \choose 3 = \Theta(n^{3/2})$

$\text{defect}(T(q+1, q)) = -2 \left( q \right) \choose 3 = -\Theta(n^{3/2})$

[Even-Zohar et al. 2014]  [Hayashi et al. 2012]
Flat torus knot $T(p,q)$

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$T(21,34)$  $T(34,21)$
Flat torus knot $T(p,q)$

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$T(21,34)$

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$$\text{defect}(T(F_{k+1}, F_k)) = \text{defect}(T(F_{k-1}, F_k))$$

$$= (F_k - 1)(F_k - 2)/3$$

$$= \Theta(n)$$
Electrical Lower Bound
medial electrical moves

[Steinitz 1916]
medial electrical moves

[Steinitz 1916]
delete  contract

minor
minor

delete

contract
minor

delete  contract

smoothing
Lemma: Reducing any connected proper minor of $G$ requires strictly fewer electrical moves than reducing $G$.

[Truemper 1989, Gitler 1991]
Lemma: The minimum \# electrical moves to reduce $G$ is \textit{at least} the minimum \# homotopy moves to simplify $G^\times$.

Proof: Replace the first $2 \rightarrow 1$ move with a $2 \rightarrow 0$ move, then apply the minor lemma and induction.

[Noble Welsh 2000]
Lemma: The minimum \# electrical moves to reduce $G$ is \textit{at least} the minimum \# homotopy moves to simplify $G^\times$.

Corollary: For all $k$, the $k \times (2k - 1)$ cylindrical grid graph requires $\Omega(k^3) = \Omega(n^{3/2})$ electrical moves to reduce.
**Lemma:** The minimum number of electrical moves to reduce $G$ is at least the minimum number of homotopy moves to simplify $G^\times$.

**Corollary:** For all $k$, the $k \times (2k - 1)$ cylindrical grid graph requires $\Omega(k^3) = \Omega(n^{3/2})$ electrical moves to reduce.

▷ **Proof:** Its medial graph is $T(2k, 2k - 1)$. 
Electrical Upper Bound

DANGER
HARD HAT AREA
We can’t modify our homotopy algorithm directly—it uses $0 \rightarrow 2$ moves, which have no electrical equivalent.
Conjecture: Every **loose tangle** either has an empty loop, has an empty bigon, or admits a 3→3 move that reduces the potential of the tangle.

- This would imply an $O(n^{3/2})$-move electrical reduction algorithm.
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- This would imply an $O(n^{3/2})$-move electrical reduction algorithm.

- Unfortunately, this conjecture is false!
**Conjecture:** Every 4-regular plane graph has either an empty loop or a bigon containing $O(n^{1/2})$ faces.

- This would also imply an $O(n^{3/2})$-move electrical reduction algorithm.
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- This would also imply an $O(n^{3/2})$-move electrical reduction algorithm.

- Unfortunately, this conjecture is also false!
Every loop and every bigon in the “Fibonacci cube” contains $\Omega(n)$ faces.
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Every loop and every bigon in the “Fibonacci cube” contains $\Omega(n)$ faces.
Lemma: Every loose tangle with at most $3n^{1/2}$ open strands contains either an empty loop or a minimal bigon with perimeter at most $O(n^{1/2})$.

Proof: Complicated case analysis and parameter balancing
Consider a minimal bigon $\beta$ with area $A$ and perimeter $P$.

- We can reduce $\beta$ to a centipede using $A - P/2$ moves, which reduce potential by at least $A/2 - P/4$.

- Then we can empty $\beta$ with $P/2 - 1$ additional moves.
Electrical Reduction Algorithm

- While the graph is not reduced
  - Find a useful tangle $T$ in the medial graph
  - While $T$ is not tight
    - Find a minimal bigon $\beta$ in $T$ with small perimeter
    - Shrink $\beta$ to a centipede
    - Empty the centipede
    - Remove the empty bigon
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$m \leq n$ vertices
$\leq m^{1/2}$ open strands
max depth $O(m^{1/2})$
potential $O(m^{3/2})$
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charge to potential
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- Max depth $O(m^{1/2})$
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- Charge to perimeter
- Charge to potential
- Charge to vertices of $T$
- Charge to deleted vertex
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$O(n^{1/2})$ charged to each vertex, so $O(n^{3/2})$ moves total.
Open questions

- Can we find a reducing sequence of $O(n^{3/2})$ electrical transformations \textit{in $O(n^{3/2})$ time}?
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- Is this stuff actually useful for anything?
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- What about planar graphs \textit{with terminals}?  
- What about \textit{non-planar} graphs?  
- Is this stuff actually \textit{useful} for anything?
Thank you!