
The homework will be graded by distributing all the solutions to each problem to one student. In order to make this possible, please start each numbered problem on a new page. If a single problem requires more than one page, staple these pages together, but do *not* staple the whole solution set together. Finally, make sure your name appears on each of your solutions.

You can use books, journal articles, notes, the web, other students, faculty, or flying monkeys to help solve these problems, but please cite any source that you use!

*Stars indicate problems whose answers I don't know. This does not necessarily imply that the problem is open, or even difficult— try them anyway!

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1. What is the *exact* randomized time complexity of the breakpoint problem described in the first lecture, as a function of n? *[Hint: An asymptotic bound of* $\Theta(\log n)$ *is easy.]*

2. Let $\prec$ be a partial order on a finite set X, such that two elements $x, y \in X$ are incomparable. Prove that there are two linear extensions of $\prec$ such that the rank of x in the first ordering is equal to the rank of y in the second ordering.

3. Let $U(n, k)$ denote the minimum number of comparisons needed to find the k largest elements, in sorted order, from an n-element set.

   (a) Prove that $U(n, k) \geq V(n, k)$. *[Hint: Be careful.]*

   (b) By describing an algorithm, give the best upper bound you can for $U(n, k)$.

   (c) Using a leaf-counting argument, prove that $U(n, k) \geq n - k + \lg \left\lceil \frac{n!}{(n-k+1)!} \right\rceil$

4. Suppose you want to determine the largest number in an n-element set $X = \{x_1, x_2, \ldots, x_n\}$, where each element x_i is an integer between 1 and $2^m - 1$. Describe an algorithm that solves this problem in $O(n + m)$ steps, where at each step, your algorithm compares one of the elements x_i with a *constant*. In particular, your algorithm must never actually compare two elements of X! *[Hint: Construct and maintain a nested set of 'pinning intervals' for the numbers that you have not yet removed from consideration, where each interval but the largest is either the upper half or lower half of the next larger block.]*

*5. UIUC has just finished constructing the new Reingold Building, the tallest structure on campus. In order to determine how much insurance to buy, the administration needs to determine the highest safe floor in the building. A floor is consdered *safe* if a student can fall from a window on that floor and survive; if the student dies, the floor is considered *unsafe*. The same floors are safe for every student, and any floor that is higher than an unsafe floor is also unsafe. The only way to determine whether a floor is safe is for a student 'volunteer' to jump out of a window on that floor. The University wants to minimize the number of jumps; however, there are only a handful of student 'volunteers' available.

   Let $J(n, s)$ denote the minimum number of jumps required to determine the highest safe floor in an n-story building, using at most s student volunteers. Derive the tightest upper and lower bounds you can for this function. *[Hint: It may help to consider one or both inverse functions* $N(j, s)$ *and* $S(n, j)$*. How should these be defined?]*