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*Stars indicate problems whose answers I don't know. This does not necessarily imply that the problem is open, or even difficult—try them anyway! Bigger stars indicate problems I think are harder and/or more likely to be open.

As a standing rule, if you publish anything as a result of work in this class, you automatically get an A+.

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1. Consider the AND-OR tree function $F_k : \{0,1\}^{2^k} \rightarrow \{0,1\}$ introduced in lecture (notes). This question asks you to derive bounds on the deterministic, certificate, and randomized complexities of this function.

   (a) Prove that $D(F_k) = 2^k$

   (b) Prove that $C(F_k) = 2^{\lfloor k/2 \rfloor}$

   (c) Prove that $R(F_k) \leq 2\left(\frac{1+\sqrt{33}}{4}\right)^{k-1}$ *[Hint: The running time should suggest a recurrence—think annihilators or generating functions. Find an algorithm that fits that recurrence!]*

2. A special case of a string property is the presence of a particular substring. We say that a binary string s is evasive for a particular value of n if the property of an n-bit string containing s as a substring is evasive. For example, the "n-card Monte" argument shows that the string 1 is evasive.

   (a) Prove that 111 is not evasive if $n \bmod 4 \in \{1, 2\}$. *[Hint: You need an algorithm.]*

   (b) Prove that 100 is evasive if and only if n is a multiple of 3.

   (c) Find all values of n for which 101 is evasive.

   *(d) Prove that *every* nonempty bitstring is evasive for infinitely many values of n. *[Hint: Derive a recurrence for the number of strings containing s from a DFA that accepts the strings containing s.]*

   ★(e) Prove that 0 and 1 are the only bitstrings that are evasive for all sufficiently large n.

3. Answer one of the following two questions:

   (a) A *scorpion* is an n-vertex undirected graph with three special vertices: the *sting*, the *tail*, and the *body*. The sting is connected only to the tail; the tail is connected only to the sting and the body; and the body is connected to every vertex except the sting. The rest of the vertices (the head, eyes, legs, antennae, teeth, gills, flippers, etc.) can be connected arbitrarily. Prove that the property of being a scorpion has deterministic complexity $O(n)$.

   ★(b) Everybody hates the scorpion question! Find another example of a nontrivial graph property that is not evasive, preferrably with deterministic complexity $O(n)$.

4. What is the exact deterministic complexity of determining whether an n-vertex directed graph has a *sink*, that is, a vertex with in-degree $n - 1$ and outdegree 0? Give both an algorithm and a matching lower bound.

5. Prove that planarity is an evasive graph property.