1. Let $\Delta(F)$ denote the minimum number of disjoint monochromatic minors needed to cover a 0/1-matrix $F$. Recall that the rank of a matrix is the size of the largest linearly-independent subset of rows (or equivalently, of columns).

(a) Prove that $\Delta(F) \geq \text{rank}(F)$. \textbf{[Hint: Let $\Delta_1(F)$ be the minimum number of disjoint monochromatic minors needed to cover only the 1s in $F$. Prove by induction on $\Delta_1(F)$ that $\Delta_1(F) \geq \text{rank}(F)$.]}

(b) Derive the tightest upper and lower bounds you can for the deterministic communication complexity of the inner product function (mod 2):

$$F(x, y) = \left( \sum_{i=1}^{n} x_i y_i \right) \mod 2.$$ 

2. Consider the exactly-$n$ function for three players.

(a) Consider the case $n = 3$. What is the exact minimum number of colors needed to color $H_3 = \{(x, y, z) \mid x + y + z = 3\}$ so that no forbidden 3-pattern is monochromatic?

*(b) Find the best upper and lower bounds you can for the minimum number of colors needed to color $H_n = \{(x, y, z) \mid x + y + z = n\}$ so that no forbidden 3-pattern is monochromatic.

*(c) Describe a communication protocol that computes the 3-player exactly-$n$ function in $O(\sqrt{\log n})$ bits.

3. Let $F$ be a random function from $\{0, 1\}^n \times \{0, 1\}^n$ to $\{0, 1\}$, for some fixed integer $n$. Prove that each of the following bounds holds with probability strictly greater than $1/2$:

(a) The largest fooling set for $F$ has size $O(n)$.

(b) $\Delta(F) = \Omega(2^n)$.

(c) Any deterministic protocol for $F$ requires $n - O(1)$ bits to be exchanged.

[Hint: How many functions have communication complexity less than $k$?]

4. (a) Let $T$ be a fixed tree on $n$ vertices. Show that the communication complexity of the following game is at most $2\lceil \lg n \rceil + 2$: Alice and Bob are each given a subtree of $T$, and they must determine whether their subtrees share a vertex.

(b) Show that the complexity of the following communication problem is exactly $d$. Alice and Bob have agreed on an arbitrary binary tree $T$ with depth $d$. When the game begins, each level of the tree is assigned a bit; Alice gets the bits assigned to all even-depth nodes, and Bob gets the bits assigned to all odd-depth nodes. Alice and Bob’s task is to compute the value of $T$, defined recursively as follows:

- If $T$ is a single leaf, then its bit is the value of $T$.
- If $T$ is an internal node labeled 0, the value of $T$ is the value of $T$’s left subtree.
- If $T$ is an internal node labeled 1, the value of $T$ is the value of $T$’s right subtree.

5. Suppose Alice and Bob are given different $n$-bit strings $x$ and $y$, and they want to compute an index $i$ such that $x_i \neq y_i$. For this problem, we’ll relax the requirement that Alice and Bob have to alternate bits. The last $\lg n$ bits broadcast must be the index $i$. There is a trivial protocol using $n + \lg n$ bits: Alice transmits $x$, and Bob replies with $i$. Describe a protocol that computes $i$ using at most $n + \log^* n$ bits. \textbf{[Hint: Alice begins by transmitting the first $n - \lg n$ bits of $x$. . . .]}