1. Let $\Delta(F)$ denote the minimum number of disjoint monochromatic minors needed to cover a 0/1-matrix F. Recall that the *rank* of a matrix is the size of the largest linearly-independent subset of rows (or equivalently, of columns).

   (a) Prove that $\boxed{\Delta(F) \geq \text{rank}(F)}$. *[Hint: Let $\Delta_1(F)$ be the minimum number of disjoint monochromatic minors needed to cover only the 1s in F. Prove by induction on $\Delta_1(F)$ that $\Delta_1(F) \geq \text{rank}(F)$.]*

   (b) Derive the tightest upper and lower bounds you can for the determinsitic communciation complexity of the inner product function (mod 2):
$$F(x, y) = \left(\sum_{i=1}^{n} x_i y_i\right) \bmod 2.$$

2. Consider the exactly-n function for three players.

   (a) Consdier the case $n = 3$. What is the exact minimum number of colors needed to color $H_3 = \{(x, y, z) \mid x + y + z = 3\}$ so that no forbidden 3-pattern is monochromatic?

   *(b) Find the best upper and lower bounds you can for the minimum number of colors needed to color $H_n = \{(x, y, z) \mid x+y+z = n\}$ so that no forbidden 3-pattern is monochromatic.

   *(c) Describe a communciation protocol that computes the 3-player exactly-n function in $O(\sqrt{\log n})$ bits.

3. Let F be a random function from $\{0, 1\}^n \times \{0, 1\}^n$ to $\{0, 1\}$, for some fixed integer n. Prove that each of the following bounds holds with probability strictly greater than 1/2:

   (a) The largest fooling set for F has size $O(n)$.

   (b) $\Delta(F) = \Omega(2^n)$.

   (c) Any deterministic protocol for F requires $n - O(1)$ bits to be exchanged.

   [Hint: How many functions have communication complexity less than k?]

4. (a) Let T be a fixed tree on n vertices. Show that the communication complexity of the following game is at most $2\lceil \lg n \rceil + 2$: Alice and Bob are each given a subtree of T, and they must determine whether their subtrees share a vertex.

   (b) Show that the complexity of the following communication problem is exactly d. Alice and Bob have agreed on an arbitrary binary tree T with depth d. When the game begins, each lebel of the tree is assigned a bit; Alice gets the bits assigned to all even-depth nodes, and Bob gets the bits assigned to all odd-depth nodes. Alice and Bob's task is to compute the *value* of T, defined recursively as follows:

   - If T is a single leaf, then its bit is the value of T.
   - If T is an internal node labeled 0, the value of T is the value of T's left subtree.
   - If T is an internal node labeled 1, the value of T is the value of T's right subtree.

5. Suppose Alice and Bob are given *different* n-bit strings x and y, and they want to compute an index i such that $x_i \neq y_i$. For this problem, we'll relax the requirement that Alice and Bob have to alternate bits. The last $\lg n$ bits broadcast must be the index i. There is a trivial protocol using $n + \lg n$ bits: Alice transmits x, and Bob replies with i. Describe a protocol that computes i using at most $\boldsymbol{n + \log^* n}$ bits. *[Hint: Alice begins by transmitting the first $n - \lg n$ bits of $x \ldots$.]*