CS 497: Computational Geometry	Spring 2002
Lecture 3 — January 22	
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This course is about "computational geometry", but those terms haven't been defined yet. Today's lecture leaves "computational" aside, and focuses on defining "geometry".

3.1 Geometry?

3.1.1 The kind of geometry we won't do: Euclidean geometry

Euclidean geometry is basically its five postulates and some rules of inference, so is basically just logic. Hilbert claimed that geometric objects don't really correspond to anything; if we were to replace "point" by "table", "line" by "chair", "plane" by "beer mug" everywhere we encountered them in the theory, nothing would be changed.

3.1.2 The kind of geometry we will be doing

Klein claimed that geometry is studying the properties of **space** that are **invariant** under a **group** of transformations.

Example. In Euclidean plane geometry, the space is $\mathbb{R}^2$, the transformations are translations, rotations, scalings and reflections (**similarities**), and the convex hull of a set of points, for example, is invariant under similarities.

We care about transformations because it is useful to have several different "pictures" of the same problem. This doesn't only apply to the plane; it is true in higher dimensions as well.

3.2 Transformations

(We ignore reflections right now. They can be expressed in terms of other things anyway).

We have the following picture:

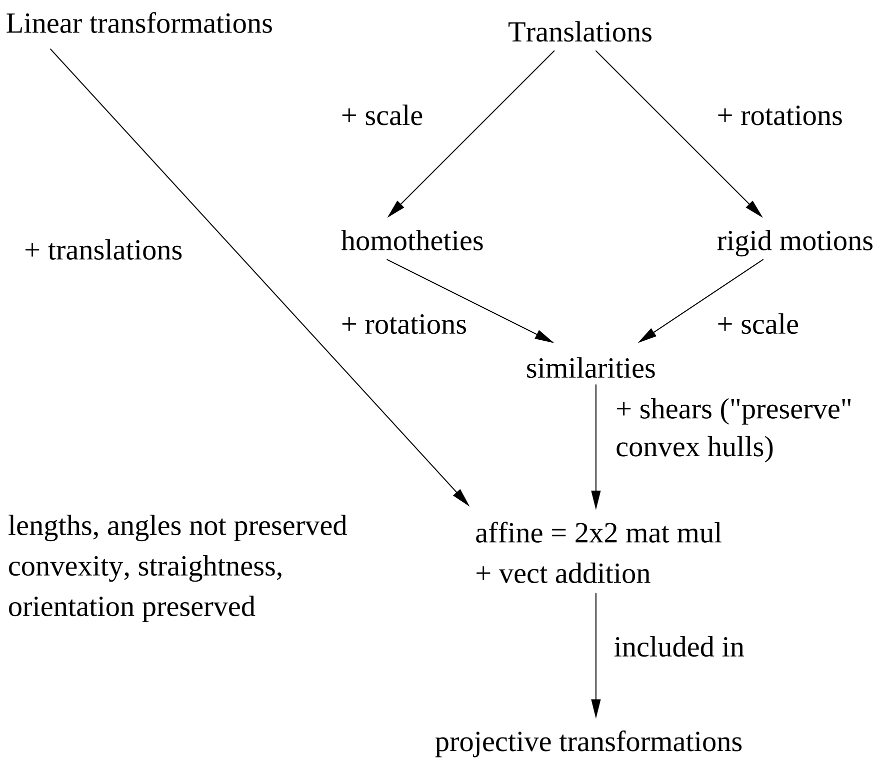

Figure 3.1. Relationships among groups of transformations

3.3 Projective transformations

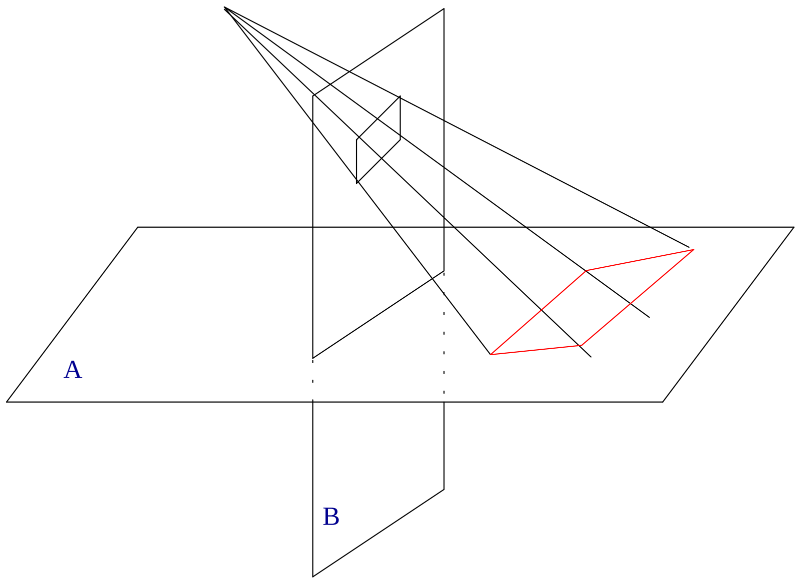

Figure 3.2. A projective transformation from plane B to plane A

3.3.1 The real projective plane

$\mathbb{R}P^2$ is:

- The sphere S^2 with antipodal points identified, or, equivalently,
- $\mathbb{R}^2$ together with the **line at** ∞, or, equivalently,
- The set of lines through the origin in $\mathbb{R}^3$.

So the usual plane can be embedded into $\mathbb{R}P^2$ since it is a subspace of it. Also, anything in $\mathbb{R}P^2$ can be projected back onto some plane.

A line in $\mathbb{R}P^2$ is a plane through the origin in $\mathbb{R}^3$, or equivalently, the equator of S^2. The line at ∞ is the xy-plane.

3.3.2 Definition of projective transformation

A projective transformation is a homogeneous linear transformation of $\mathbb{R}^3$.

Here Jeff wrote on the board: a projective transformation of $\mathbb{R}^2$ is a linear transformation of $\mathbb{R}^3$. I think it should be "homogeneous linear transformation", which is what was captured by this picture that he drew:

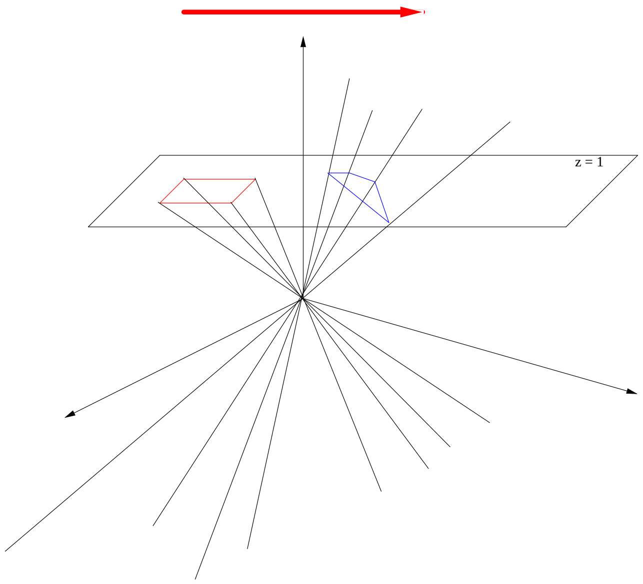

Figure 3.3. A linear transformation in $\mathbb{R}P^2$.

Artist's perspective: The line at ∞ is the horizon if you stand on a plane and look in the distance. A projective transformation (possibly) makes the line at ∞ not be at ∞ anymore, and some other line gets bumped out of the picture towards ∞.

3.4 The convex hull problem and projective transformations

When drawing a picture of a projective transformation, that looks like it's in $\mathbb{R}^2$ but that is really in $\mathbb{R}P^2$, it is important to specify what line goes to ∞ and where the line at ∞ gets mapped to.

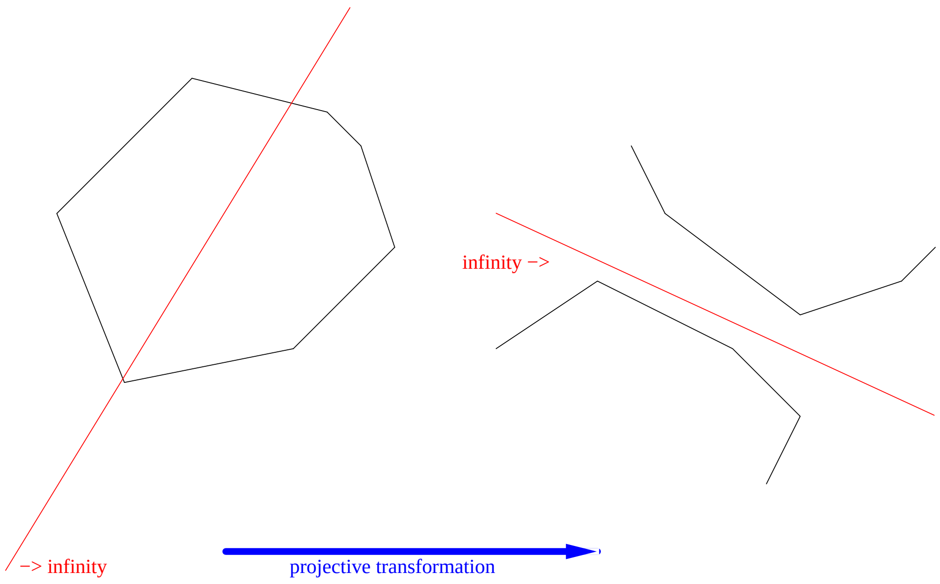

Figure 3.4. A convex hull before and after a projective transformation

3.4.1 Another way to think of the convex hull problem

There's a bunch of nails on a wall, and a ruler somewhere. Where can the ruler (rigidly) move around? Solution: projectivize, send the ruler to the line at ∞, and it becomes the convex hull problem. So the convex hull problem is: what is the area of the plane that is inaccessible to some line given a set of points?

3.4.2 Do projective transformations preserve orientation?

In order for this to be true, we have to put an orientation on $\mathbb{R}P^2$. But it is a non-orientable manifold. So we need to de-identify antipodal points, and use for example the right-hand rule to determine orientation. However, that's ugly, so we don't worry about it. See Stolfi for an implementation of this.

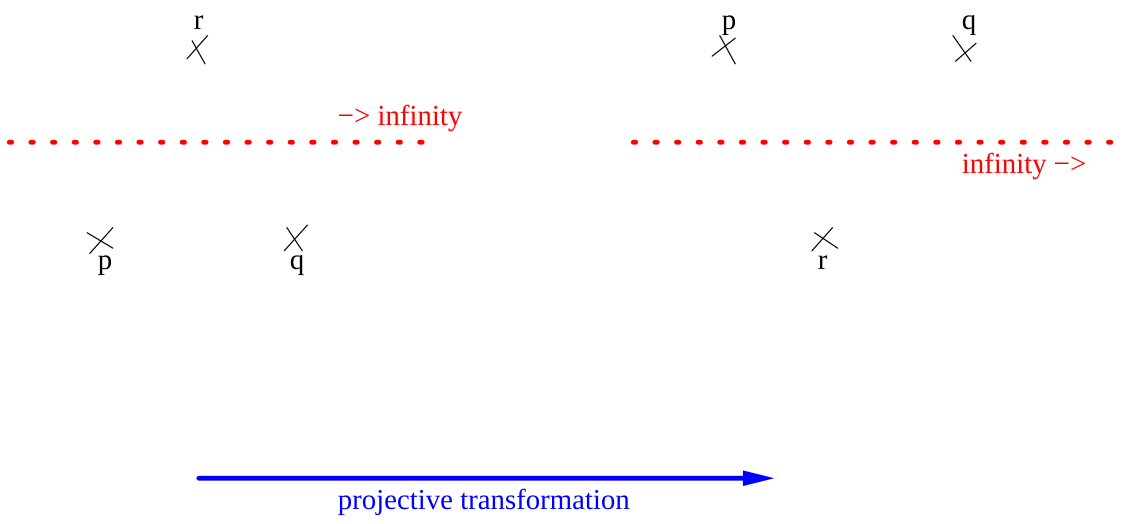

Figure 3.5. Do projective transformations preserve orientation?

3.5 Duality

Theorem 3.1 (Fundamental (Meta-) Theorem of Projective Geometry). *Any statement about Points and Lines is equivalent to the corresponding statement about Lines and Points.*

(i.e., s/lines/points/g, s/points/lines/g).

There are many different duality maps, ie converting from points to lines and from lines to points. Here are two.

3.5.1 The First Duality Map

Primal	**Dual**
Point	line
(a, b)	$y = ax - b$
line at ∞	point at ∞
vertical line	point at ∞

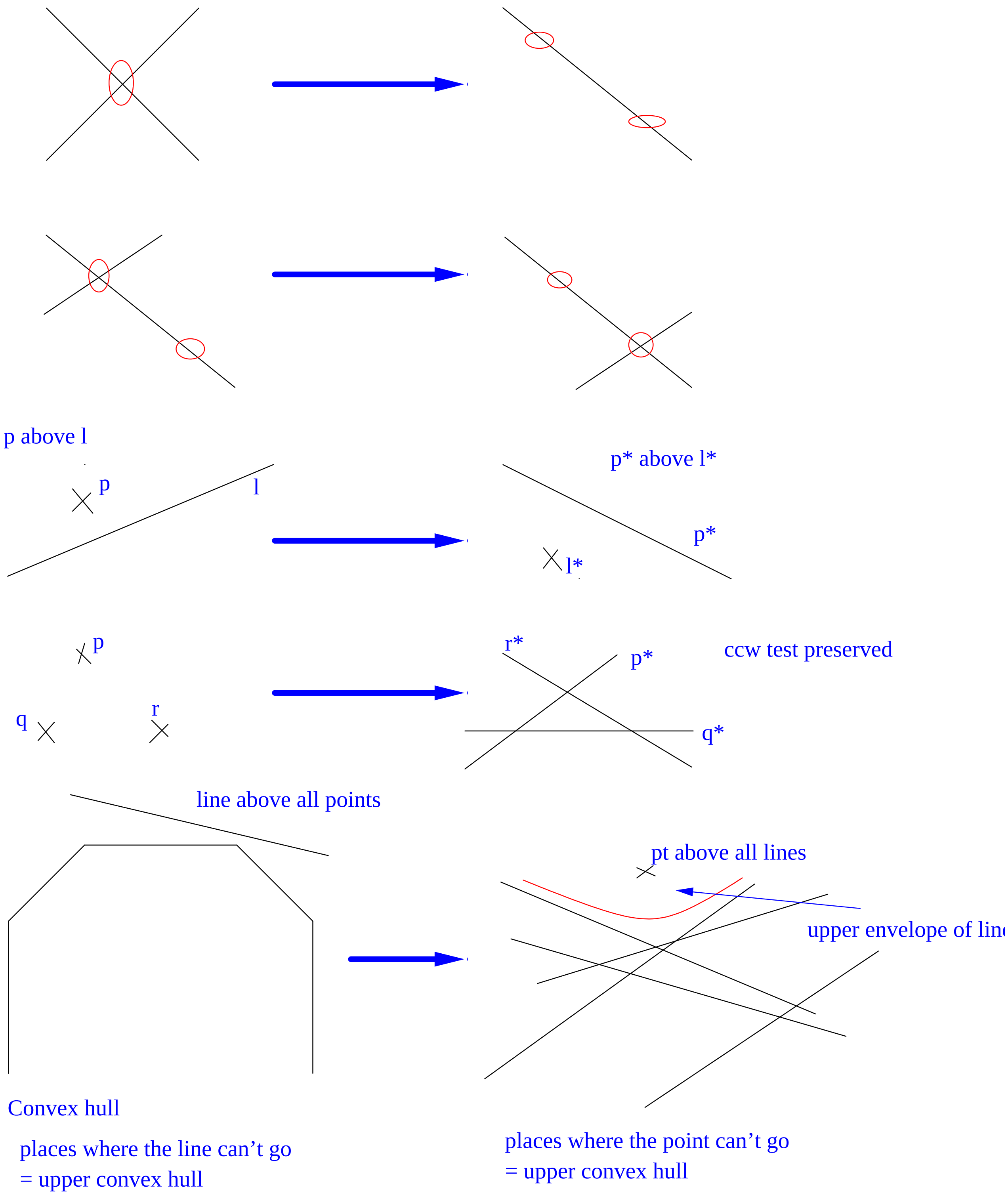

Figure 3.6. Some dualities

3.5.2 The Second Duality Map

Primal	**Dual**
(a, b)	$ax + by = 1$
p closer to the origin than l	p^* further away from origin than l^*.
line through origin	point at ∞
origin	line at ∞

Vanum est vobis ante lucem surgere.
(Nisi Dominus)