CS 497 JE: Computational Geometry	**Spring 2002**

Lecture 3 — Jan. 24, 2002

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This lecture covered two topics: the summary of graph transformation, which reviewed the materials covered by the last lecture and summarized them, and Voronoi Diagrams.

3.1 Graph Transformation

3.1.1 Rigid Motion

Rigid motion consists of translation and rotation. Translation is a transformation consisting of a constant offset with no Rotation or distortion. In n-D Euclidean Space, a translation may be specified simply as a vector $\mathbf{v}$ giving the offset in each of the n coordinates. Let $\mathbf{x} = \langle x_1, x_2, ..., x_n \rangle$ be a point in the n-D Euclidean space and $\mathbf{x}'$ be the translated point. A translation moving $\mathbf{x}$ by $\mathbf{v}$ leads to that $\mathbf{x}' = \mathbf{x} + \mathbf{v}$.

Rotation rotates the turning of an object or coordinate system by an angle about a fixed point.

When discussing a Rotation, there are two possible conventions: rotation of the axes and rotation of the object relative to fixed axes. Consider rotating a curve on a $\mathbb{R}^2$ plane as an example. In $\mathbb{R}^2$, let a curve be rotated by a clockwise angle θ, eso that the original axes of the curve are $\mathbf{x}$ and $\mathbf{y}$, and the new axes of the curve are $\mathbf{x}'$ and $\mathbf{y}'$.

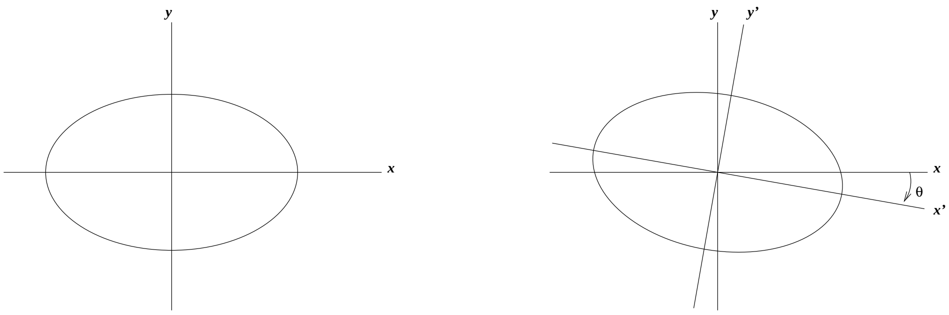

Figure 3.1. Rotate a curve by angle θ

The matrix transforming the original curve to the rotated curve, referred to the original $\mathbf{x}$ and $\mathbf{y}$ axes, is

$$\mathsf{R}_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

i.e.,

$$\mathbf{x} = \mathsf{R}_\theta \mathbf{x}'.$$

Therefore, the transformation by a rigid motion which moves the graph by $\mathbf{v}$ offset and rotates θ angle clockwise can be presented by

$$\mathbf{x} = \mathsf{R}_\theta \mathbf{x}' + \mathbf{v}.$$

It is easy to see that the distance between any two points are preserved according to the above equation. As a result, rigid motion also preserves the angle between any pair of two lines. Moreover, lines in the original graph are still lines in the transformed graph. The transformation also preserves the area of an region. Unit circles whose radius are 1 and unit squares whose sides are all 1 are also preserved. So does simple polygon and convex polygon.

3.1.2 Similarity

Similarity consists of rigid motion and scaling. Scaling increases a plane graph's linear dimensions by a scale factor s. By scaling, the distance between two points are increased by s, however, the distance ratio are maintained. For any three points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, the distance ratio is defined as

$$\frac{\mid \mathbf{y}x \mid}{\mid \mathbf{y}z \mid}.$$

Therefore, after the similarity transformation, the distance ratio of these three points $\mathbf{x}'$, $\mathbf{y}'$, and $\mathbf{z}'$ is as same as that of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. In other words,

$$\frac{\mid \mathbf{y}x \mid}{\mid \mathbf{y}z \mid} = \frac{\mid \mathbf{y}'\mathbf{x}' \mid}{\mid \mathbf{y}'\mathbf{z}' \mid}.$$

Consequently, scaling increases the perimeter $p' \rightarrow s\,p$ and and the area $A' \rightarrow s^2 A$. However, for any two regions, the area ratio is preserved. As for the shape of the graph, lines, circles, and squares are maintained, however, unit circles and unit squares are no longer the same.

3.1.3 Affine

Affine is the translation plus the linear transformation. Specifically, an affine transformation of $\mathbb{R}^n$ is a map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$F(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{v}$$

for all $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, where A is a linear transformation of $\mathbb{R}^n$. Naturally, rigid motion and similarity are all affine transformations, as are their combinations.

As a result, area ratio is preserved and lines are preserved. However, circles may be changed to ellipses because of the linear transformation matrix A. Squares may be transformed into parallelograms.

Any affine transformation is the transformation of points and lines in one plane onto another plane by connecting corresponding points on the two planes with Parallel lines. This can be visualized as shining a (point) light source (located at infinity) through a translucent sheet of paper and making an image of whatever is drawn on it on a second sheet of paper.

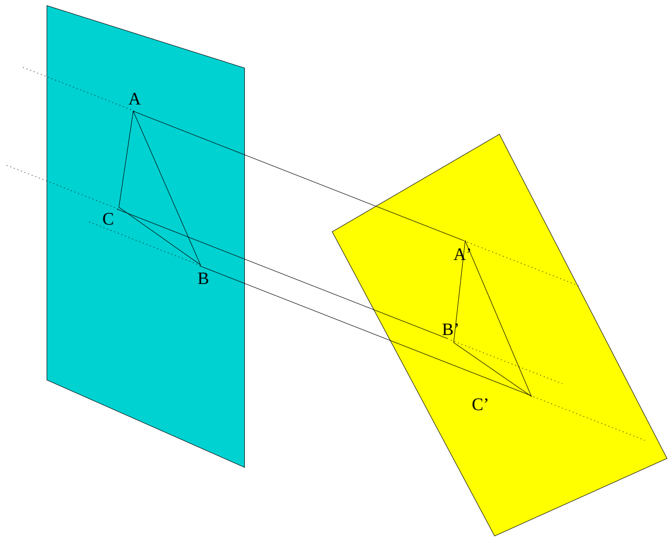

Figure 3.2. An affine transformation

3.1.4 Projective

A projective transformation can be conducted by the operation of two perspective transformations. A reflective transformation is the operation of exchanging all points with their mirror images (i.e., reflections in a mirror). Figure 3.3 shows an example. In Figure 3.3, p is the mirror point. All points on the original plane are mirrored to another plane in terms of the mirror point p.

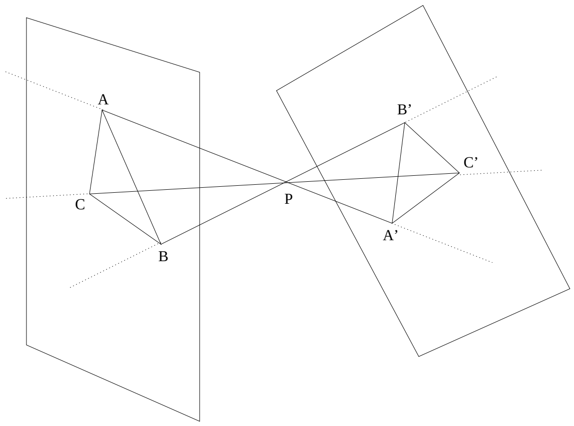

Figure 3.3. A perspective transformation

A projective transformation preserves the cross ratio among any four points. The cross ration for four points A, B, C, and D is defined as

$$\frac{\mid AC \mid \mid BD \mid}{\mid AD \mid \mid BC \mid}.$$

The cross ratio remains constant.

Projective transformations preserve lines on the plane, but do not preserve circles. For instance, when a projective map is applied to a circle w.r.t. an (infinity) line crossing the center, the graph becomes a hyperbola. On the other hand, the graph becomes a parabola if the infinite line is tangent to or outside the circle. Any type of nondegenerate conic (ellipse, parabola, hyperbola) can be projectively mapped to any other.

Furthermore, parallelograms are transformed to quadrilaterals. As for the convex polygon, it is still a convex w.r.t. to some particular infinity line.

The geometry dealing with the properties and invariants of geometric figures under Projection is called the projective geometry. The important property arising in projective geometry is the duality principle, which states that a duality exists between theorems such as Pascal's theorem and Brianchon's theorem which allows one to be instantly transformed into the other. More generally, all the propositions in projective geometry occur in dual pairs, which have the property that, starting from either proposition of a pair, the other can be immediately inferred by interchanging the parts played by the words "point" and "line." The duality shall be discussed later in this course.

3.1.5 Conformal

A conformal transformation is a transformation $w = f(z)$ that preserves angles. We are typically interested in *Möbius* transformations, which also preserve circles. (In higher dimensions, Möbius transformations are the only conformal maps, but this is not true in the plane.)

A Möbius transformation can be conducted by two inversions transformations. Inversion is the process of transforming points P to a corresponding set of points P' known as their inverse points. Two points p and p' are said to be inverses with respect to an inversion circle having inversion center $o = (x_0, y_0)$ and inversion radius 1 if p' is on the vector $\overrightarrow{op}$ and $\mid op' \mid = \frac{1}{|op|}$. (Notes: the radius of the inversion circle is not necessary to be 1. If the radius is not 1, $\mid op' \mid \times \mid op \mid$ must be r^2 where r is the radius of the inversion circle.) The analogous notation of inversion can be carried to in 3-dimensional space with respect to an inversion sphere. Stereographic projection, which will be discussed later, is a special case of such transformation.

Treating lines as circles of infinite radius, all circles invert to circles. Furthermore, any two nonintersecting circles can be inverted into concentric circles by taking the inversion center at one of the two so-called limiting points of the two circles, and any two circles can be inverted into themselves or into two equal circles. Performing the inversion transforms twice completes the conformal transformation and angles are preserved. Although lines are transformed into circles, the angle of the intersection of two circles is as same as that of these two original lines as shown in Figure 3.5.

Stereographic projection projects points on a sphere onto a plane. A Stereographic projection obtained by projecting points A on the surface of sphere from the sphere's north pole to point A' in a plane tangent to the south pole. In such a projection, great circles are mapped to circles. If we consider the diameter of the sphere is the radius of the inversion sphere, stereographic projection is a special case of the 3-D inversion transformation. However, ellipses, parabolas, hyperbolas are transformed some wired graph.

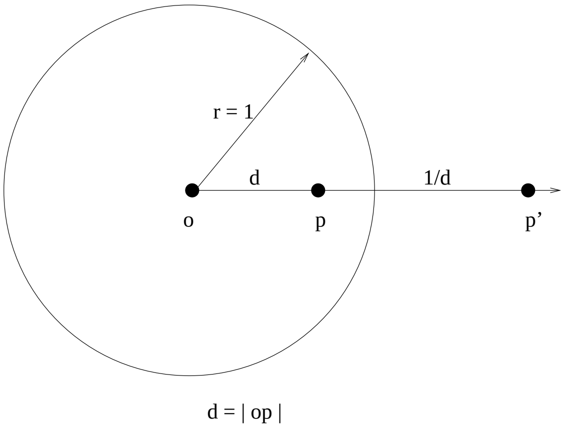

Figure 3.4. Inversion transformation

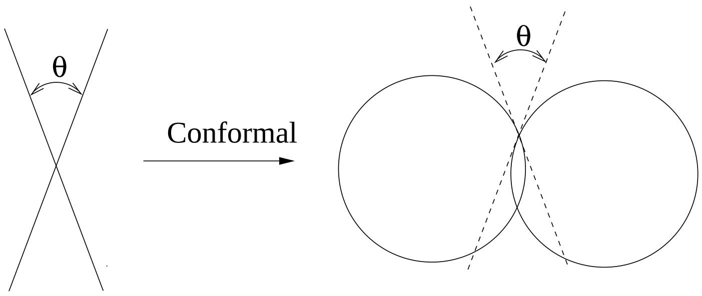

Figure 3.5. Angles for conformal transformation

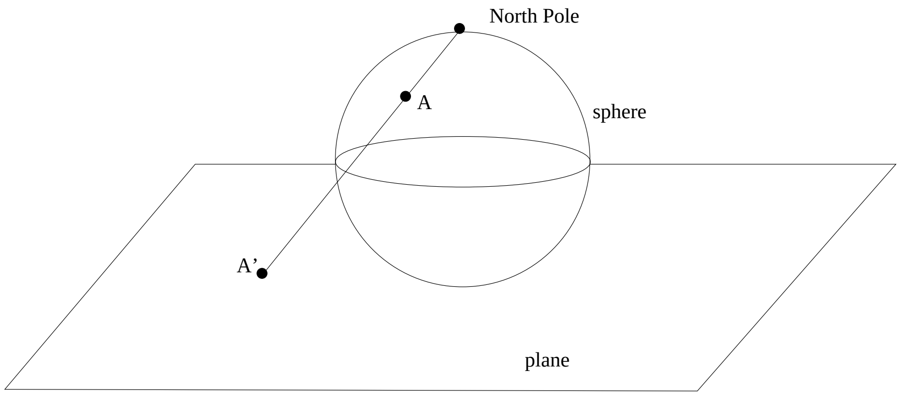

Figure 3.6. Stereographic Projection

Conformal mapping is extremely important in complex analysis. On the complex plane, conformal transformation can be carried out by a function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = \frac{az+b}{cz+d}$ where z is a complex and a, b, c, d are four numbers. Similarity, which can be presented by $z \to az + b$, is a special case of conformal transformation where $c = 1$ and $d = 0$.

For a circle on a sphere, a conformal transforms it into another circle onto another sphere.

Projective map on $\mathbb{RP}^3$ preserving some sphere is equal to the conformal mapping.

Another way of saying conformal transformation is using hyperbolic geometry. Hyperbolic geometry satisfies all of Euclid's postulates except the parallel postulate, which is modified to read: For any infinite straight line L and any point p not on it, there are many other infinitely extending straight lines that pass through p and which do not intersect L. This property can be visualizing in the following way. Line L lies on one of two hyperboloid and point p lies on another hyperboloid. As a result, all lines pass through point p will not intersect line L as these lines are on another hyperboloid.

3.1.6 Summary

Table 3.7 summarizes the properties of these transformations.

3.2 Voronoi Diagrams

Voronoi Diagrams is a set of region on $\mathbb{R}^2$ such that the distance from any point x on the plane to point $q \in \mathbf{P}$ is less than the distance from x to any other point $p \in \mathbf{P}$. For a set of points P in the plane, Voronoi diagram $Vor(q, P)$ where $q \in P$ divides the plane to regions such that

$$Vor(q, P) = \{x \in \mathbb{R}^2 \mid d(x, q) \leq \min_{p \in P} d(x, p)\}.$$

Figure 3.7 shows one example.

Particularly, if $q \in \mathcal{CH}(P)$, $Vor(q, P)$ will be unbounded. Otherwise, it will be a convex. One Voronoi region can be computed in $O(n \log n)$ using any convex hull algorithm. However, computing all Voronoi diagram in this way takes $O(n^2 \log n)$ to complete. Algorithms of finding Voronoi diagram will be continued in the next lecture.

Note: most materials are from the class and some of them comes from the website of *Eric Weissten's World of Mathematics* on mathworld.wolfram.com.

Table 3.7: Summary of transformations.

	Rigid Motion	$\subseteq$	Similarity	$\subseteq$	Affine	$\subseteq$	Projective	Conformal(incl. Similarity)
Definition	Translation + scaling		Rigid + scale		translation + linear		$2 \times$ reflective(perspective)	$2 \times$ inversion
Preserved Properties	distance bwt 2 points		distance ratio		none		none	none
	Angle bwt 2 lines		(yes)		(no)		(no)	(yes)
	Area		Area Ratio		Area ratio			
	Lines		Lines		Lines		"Lines"	
	Unit circles		Circles		Ellipses, parabolas, hyperbolas			
	Unit squares		Squares		Parallelograms			
	Simple Polygon		Simple Polygon					
	Convex Polygon		Convex Polygon					

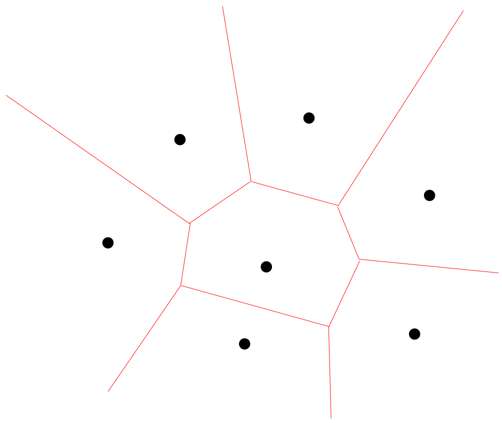

Figure 3.7. Voronoi Diagram