When you come to a fork in the road, take it.
— Yogi Berra

2 Backtracking

In this lecture, I want to describe another recursive algorithm strategy called backtracking. A backtracking algorithm tries to build a solution to a computational problem incrementally. Whenever the algorithm needs to decide between two alternatives to the next component of the solution, it simply tries both options recursively.

2.1 $n$-Queens

The prototypical backtracking problem is the classical $n$-Queens Problem, first proposed by German chess enthusiast Max Bezzel in 1848 for the standard $8 \times 8$ board, and both solved and generalized to larger boards by Franz Nauck in 1850. The problem is to place $n$ queens on an $n \times n$ chessboard, so that no two queens can attack each other. For readers not familiar with the rules of chess, this means that no two queens are in the same row, column, or diagonal.

![8 Queens Solution](image_url)

One solution to the 8 queens problem, represented by the array $[4,7,3,8,2,5,1,6]$

Obviously, in any solution to the $n$-Queens problem, there is exactly one queen in each column. So we will represent our possible solutions using an array $Q[1..n]$, where $Q[i]$ indicates the square in column $i$ that contains a queen, or 0 if no queen has yet been placed in column $i$. To find a solution, we will put queens on the board row by row, starting at the top. A partial solution is an array $Q[1..n]$ whose first $r - 1$ entries are positive and whose last $n - r + 1$ entries are all zeros, for some integer $r$.

The following recursive algorithm recursively enumerates all complete $n$-queens solutions that are consistent with a given partial solution. The input parameter $r$ is the first empty row. Thus, to compute all $n$-queens solutions with no restrictions, we would call RecursiveNQueens($Q[1..n], 1$).
Like most recursive algorithms, the execution of a backtracking algorithm can be illustrated using a recursion tree. The root of the recursion tree corresponds to the original invocation of the algorithm; edges in the tree correspond to recursive calls. A path from the root down to any node shows the history of a partial solution to the $n$-Queens problem, as queens are added to successive rows. The leaves correspond to partial solutions that cannot be extended, either because there is already a queen on every row, or because every position in the next empty row is in the same row, column, or diagonal as an existing queen.

The complete recursion tree for our algorithm for the 4-queens problem.

### 2.2 Subset Sum

Let’s consider a more complicated problem, called **Subset Sum**: Given a set $X$ of positive integers and target integer $T$, is there a subset of elements in $X$ that add up to $T$? Notice that there can be more than one such subset. For example, if $X = \{8, 6, 7, 5, 3, 10, 9\}$ and $T = 15$, the answer is $\text{TRUE}$, thanks to
the subsets \{8, 7\} or \{7, 5, 3\} or \{6, 9\} or \{5, 10\}. On the other hand, if \(X = \{11, 6, 5, 1, 7, 13, 12\}\) and \(T = 15\), the answer is False.

There are two trivial cases. If the target value \(T\) is zero, then we can immediately return True, because the elements of the empty set add up to zero.\(^1\) On the other hand, if \(T < 0\), or if \(T \neq 0\) but the set \(X\) is empty, then we can immediately return False.

For the general case, consider an arbitrary element \(x \in X\). (We've already handled the case where \(X\) is empty.) There is a subset of \(X\) that sums to \(T\) if and only if one of the following statements is true:

- There is a subset of \(X\) that includes \(x\) and whose sum is \(T\).
- There is a subset of \(X\) that excludes \(x\) and whose sum is \(T\).

In the first case, there must be a subset of \(X\) \(\setminus\{x\}\) that sums to \(T - x\); in the second case, there must be a subset of \(X\) \(\setminus\{x\}\) that sums to \(T\). So we can solve \(\text{SubsetSum}(X, T)\) by reducing it to two simpler instances: \(\text{SubsetSum}(X \setminus \{x\}, T - x)\) and \(\text{SubsetSum}(X \setminus \{x\}, T)\). Here's how the resulting recursive algorithm might look if \(X\) is stored in an array.

```python
SubsetSum(X[1..n], T):
    if T = 0
        return True
    else if T < 0 or n = 0
        return False
    else
        return SubsetSum(X[2..n], T) or SubsetSum(X[2..n], T - X[1])
```

Proving this algorithm correct is a straightforward exercise in induction. If \(T = 0\), then the elements of the empty subset sum to \(T\), so True is the correct output. Otherwise, if \(T\) is negative or the set \(X\) is empty, then no subset of \(X\) sums to \(T\), so False is the correct output. Otherwise, if there is a subset that sums to \(T\), then either it contains \(X[1]\) or it doesn't, and the Recursion Fairy correctly checks for each of those possibilities. Done.

The running time \(T(n)\) clearly satisfies the recurrence \(T(n) \leq 2T(n - 1) + O(1)\), so the running time is \(T(n) = O(2^n)\) by the annihilator method.

Here is a similar recursive algorithm that actually constructs a subset of \(X\) that sums to \(T\), if one exists. This algorithm also runs in \(O(2^n)\) time.

```python
ConstructSubset(X[1..n], T):
    if T = 0
        return ∅
    if T < 0 or n = 0
        return None
    Y ← ConstructSubset(X[2..n], T)
    if Y ≠ None
        return Y
    Y ← ConstructSubset(X[2..n], T - X[1])
    if Y ≠ None
        return Y ∪ {X[1]}
    return None
```

These two algorithms are examples of a general algorithmic technique called backtracking. You can imagine the algorithm searching through a binary tree of recursive possibilities like a maze, trying to

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\(^1\)There's no base case like the vacuous base case!
find a hidden treasure ($T = 0$), and backtracking whenever it reaches a dead end ($T < 0$ or $n = 0$). For some problems, there are tricks that allow the recursive algorithm to recognize some branches as dead ends without exploring them directly, thereby speeding up the algorithm; two such problems are described later in these notes. Alas, \textsc{SubsetSum} is not one of those problems; in the worst case, our algorithm explicitly considers every subset of $X$.\footnote{Indeed, because \textsc{SubsetSum} is NP-hard, there is almost certainly no way to solve it in subexponential time.}

### 2.3 Longest Increasing Subsequence

Now suppose we are given a sequence of integers, and we want to find the longest subsequence whose elements are in increasing order. More concretely, the input is an array $A[1..n]$ of integers, and we want to find the longest sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $A[i_j] < A[i_{j+1}]$ for all $j$.

To derive a recursive algorithm for this problem, we start with a recursive definition of the kinds of objects we're playing with: sequences and subsequences.

- A sequence of integers is either empty or an integer followed by a sequence of integers.

This definition suggests the following strategy for devising a recursive algorithm. If the input sequence is empty, there's nothing to do. Otherwise, we should figure out what to do with the first element of the input sequence, and let the Recursion Fairy take care of everything else. We can formalize this strategy somewhat by giving a recursive definition of subsequence (using array notation to represent sequences):

- The only subsequence of the empty sequence is the empty sequence.

We're not just looking for just any subsequence, but a longest subsequence with the property that elements are in increasing order. So let's try to add those two conditions to our definition. (I'll omit the familiar vacuous base case.)


This definition is correct, but it's not quite recursive—we're defining 'longest increasing subsequence' in terms of the different 'longest increasing subsequence with elements larger than $x$', which we haven't properly defined yet. Fortunately, this second object has a very similar recursive definition. (Again, I'm omitting the vacuous base case.)

- If $A[1] \leq x$, the LIS of $A[1..n]$ with elements larger than $x$ is the LIS of $A[2..n]$ with elements larger than $x$.
The longest increasing subsequence without restrictions can now be redefined as the longest increasing subsequence with elements larger than $-\infty$. Rewriting this recursive definition into pseudocode gives us the following recursive algorithm.

```
LIS(A[1..n]):
    return LISBIGGER(-\infty,A[1..n])
```

The running time of this algorithm satisfies the recurrence $T(n) \leq 2T(n-1) + O(1)$, so as usual the annihilator method implies that $T(n) = O(2^n)$. We really shouldn’t be surprised by this running time; in the worst case, the algorithm examines each of the $2^n$ subsequences of the input array.

The following alternative strategy avoids defining a new object with the ‘larger than $x$’ constraint. We still only have to decide whether to include or exclude the first element $A[1]$. We consider the case where $A[1]$ is excluded exactly the same way, but to consider the case where $A[1]$ is included, we remove any elements of $A[2..n]$ that are larger than $A[1]$ before we recurse. This new strategy gives us the following algorithm:

```
FILTER(A[1..n], x):
    j ← 1
    for i ← 1 to n
        if A[i] > x
            B[j] ← A[i]; j ← j + 1
    return B[1..j]
```

The FILTER subroutine clearly runs in $O(n)$ time, so the running time of LIS satisfies the recurrence $T(n) \leq 2T(n-1) + O(n)$, which solves to $T(n) \leq O(2^n)$ by the annihilator method. This upper bound pessimistically assumes that FILTER never actually removes any elements; indeed, if the input sequence is sorted in increasing order, this assumption is correct.

### 2.4 Optimal Binary Search Trees

Our next example combines recursive backtracking with the divide-and-conquer strategy.

Hopefully you remember that the cost of a successful search in a binary search tree is proportional to the number of ancestors of the target node. As a result, the worst-case search time is proportional to the depth of the tree. To minimize the worst-case search time, the height of the tree should be as small as possible; ideally, the tree is perfectly balanced.

In many applications of binary search trees, it is more important to minimize the total cost of several searches than to minimize the worst-case cost of a single search. If $x$ is a more ‘popular’ search target

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3 An ancestor of a node $v$ is either the node itself or an ancestor of the parent of $v$. A proper ancestor of $v$ is either the parent of $v$ or a proper ancestor of the parent of $v$. 

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we can partition the cost function into three parts as follows:

The base case for this recurrence is, as usual,

Thus, the total cost of performing all the binary searches is given by the following expression:

formally, let

A constant factors, the cost of searching for

This situation suggests the following problem. Suppose we are given a sorted array of

simple substitution gives us our recursive definition for

Now our task is to compute the tree

The base case for this recurrence is, as usual, \( n = 0 \); the cost of performing no searches in the empty tree is zero.

Now our task is to compute the tree \( T_{\text{opt}} \) that minimizes this cost function. Suppose we somehow magically knew that the root of \( T_{\text{opt}} \) is \( v_r \). Then the recursive definition of \( \text{Cost}(T, f) \) immediately implies that the left subtree \( \text{left}(T_{\text{opt}}) \) must be the optimal search tree for the keys \( A[1..r-1] \) and access frequencies \( f[1..r-1] \). Similarly, the right subtree \( \text{right}(T_{\text{opt}}) \) must be the optimal search tree for the keys \( A[r+1..n] \) and access frequencies \( f[r+1..n] \). Once we choose the correct key to store at the root, the Recursion Fairy will automatically construct the rest of the optimal tree for us. More formally, let \( \text{OptCost}(f[1..n]) \) denote the total cost of the optimal search tree for the given frequency counts. We immediately have the following recursive definition.

\[
\text{OptCost}(f[1..n]) = \min_{1 \leq r \leq n} \left\{ \text{OptCost}(f[1..r-1]) + \sum_{i=1}^{n} f[i] + \text{OptCost}(f[r+1..n]) \right\}
\]
Again, the base case is $\text{OptCost}(f[1..0]) = 0$; the best way to organize no keys, which we will plan to search zero times, is by storing them in the empty tree!

This recursive definition can be translated mechanically into a recursive algorithm, whose running time $T(n)$ satisfies the recurrence

$$T(n) = \Theta(n) + \sum_{k=1}^{n} (T(k-1) + T(n-k)).$$

The $\Theta(n)$ term comes from computing the total number of searches $\sum_{i=1}^{n} f[i]$.

Yeah, that's one ugly recurrence, but it's actually easier to solve than it looks. To transform it into a more familiar form, we regroup and collect identical terms, subtract the recurrence for $T(n-1)$ to get rid of the summation, and then regroup again.

$$T(n) = \Theta(n) + 2 \sum_{k=0}^{n-1} T(k)$$

$$T(n-1) = \Theta(n-1) + 2 \sum_{k=0}^{n-2} T(k)$$

$$T(n) - T(n-1) = \Theta(1) + 2T(n-1)$$

$$T(n) = 3T(n-1) + \Theta(1)$$

The solution $T(n) = \Theta(3^n)$ now follows from the annihilator method.

Let me emphasize that this recursive algorithm does not examine all possible binary search trees. The number of binary search trees with $n$ nodes satisfies the recurrence

$$N(n) = \sum_{r=1}^{n-1} (N(r-1) \cdot N(n-r)),$$

which has the closed-from solution $N(n) = \Theta(4^n/\sqrt{n})$. Our algorithm saves considerable time by searching independently for the optimal left and right subtrees. A full enumeration of binary search trees would consider all possible pairings of left and right subtrees; hence the product in the recurrence for $N(n)$.

**Exercises**

1. (a) Let $A[1..m]$ and $B[1..n]$ be two arbitrary arrays. A **common subsequence** of $A$ and $B$ is both a subsequence of $A$ and a subsequence of $B$. Give a simple recursive definition for the function $\text{lcs}(A,B)$, which gives the length of the longest common subsequence of $A$ and $B$.

   (b) Let $A[1..m]$ and $B[1..n]$ be two arbitrary arrays. A **common supersequence** of $A$ and $B$ is another sequence that contains both $A$ and $B$ as subsequences. Give a simple recursive definition for the function $\text{scs}(A,B)$, which gives the length of the shortest common supersequence of $A$ and $B$.

   (c) Call a sequence $X[1..n]$ oscillating if $X[i] < X[i+1]$ for all even $i$, and $X[i] > X[i+1]$ for all odd $i$. Give a simple recursive definition for the function $\text{lso}(A)$, which gives the length of the longest oscillating subsequence of an arbitrary array $A$ of integers.

   (d) Give a simple recursive definition for the function $\text{sos}(A)$, which gives the length of the shortest oscillating supersequence of an arbitrary array $A$ of integers.
(e) Call a sequence $X[1..n]$ \textit{accelerating} if $2 \cdot X[i] < X[i - 1] + X[i + 1]$ for all $i$. Give a simple recursive definition for the function $\text{lxs}(A)$, which gives the length of the longest accelerating subsequence of an arbitrary array $A$ of integers.

\textbf{For more backtracking exercises, see the next two lecture notes!}