A little and a little, collected together, become a great deal; the heap in the barn consists of single grains, and drop and drop makes an inundation.

— Saadi (1184–1291)

The trees that are slow to grow bear the best fruit.

— Molière (1622–1673)

Promote yourself but do not demote another.

— Rabbi Israel Salanter (1810–1883)

Fall is my favorite season in Los Angeles, watching the birds change color and fall from the trees.

— David Letterman

L Fibonacci Heaps

L.1 Mergeable Heaps

A mergeable heap is a data structure that stores a collection of keys and supports the following operations.

- **INSERT**: Insert a new key into a heap. This operation can also be used to create a new heap containing just one key.
- **FINDMIN**: Return the smallest key in a heap.
- **DELETEMIN**: Remove the smallest key from a heap.
- **MERGE**: Merge two heaps into one. The new heap contains all the keys that used to be in the old heaps, and the old heaps are (possibly) destroyed.

If we never had to use **DELETEMIN**, mergeable heaps would be completely trivial. Each “heap” just stores to maintain the single record (if any) with the smallest key. **INSERTS** and **MERGES** require only one comparison to decide which record to keep, so they take constant time. **FINDMIN** obviously takes constant time as well.

If we need **DELETEMIN**, but we don’t care how long it takes, we can still implement mergeable heaps so that **INSERTS**, **MERGES**, and **FINDMINS** take constant time. We store the records in a circular doubly-linked list, and keep a pointer to the minimum key. Now deleting the minimum key takes \( \Theta(n) \) time, since we have to scan the linked list to find the new smallest key.

In this lecture, I’ll describe a data structure called a **Fibonacci heap** that supports **INSERTS**, **MERGES**, and **FINDMINS** in constant time, even in the worst case, and also handles **DELETEMIN** in \( O(\log n) \) amortized time. That means that any sequence of \( n \) **INSERTS**, \( m \) **MERGES**, \( f \) **FINDMINS**, and \( d \) **DELETEMINS** takes \( O(n + m + f + d \log n) \) time.

L.2 Binomial Trees and Fibonacci Heaps

A **Fibonacci heap** is a circular doubly linked list, with a pointer to the minimum key, but the elements of the list are not single keys. Instead, we collect keys together into structures called **binomial heaps**. Binomial heaps are trees that satisfy the **heap property**—every node has a smaller key than its children—and have the following special recursive structure.

A **kth order binomial tree**, which I’ll abbreviate \( B_k \), is defined recursively. \( B_0 \) is a single node. For all \( k > 0 \), \( B_k \) consists of two copies of \( B_{k-1} \) that have been linked together, meaning that the root of one \( B_{k-1} \) has become a new child of the other root.

---

1In the earlier lecture on treaps, I called these keys priorities to distinguish them from search keys.
Binomial trees have several useful properties, which are easy to prove by induction (hint, hint).

- The root of $B_k$ has degree $k$.
- The children of the root of $B_k$ are the roots of $B_0, B_1, \ldots, B_{k-1}$.
- $B_k$ has height $k$.
- $B_k$ has $2^k$ nodes.
- $B_k$ can be obtained from $B_{k-1}$ by adding a new child to every node.
- $B_k$ has $\binom{k}{d}$ nodes at depth $d$, for all $0 \leq d \leq k$.
- $B_k$ has $2^{k-h-1}$ nodes with height $h$, for all $0 \leq h < k$, and one node (the root) with height $k$.

Although we normally don’t care in this class about the low-level details of data structures, we need to be specific about how Fibonacci heaps are actually implemented, so that we can be sure that certain operations can be performed quickly. Every node in a Fibonacci heap points to four other nodes: its parent, its ‘next’ sibling, its ‘previous’ sibling, and one of its children. The sibling pointers are used to join the roots together into a circular doubly-linked root list. In each binomial tree, the children of each node are also joined into a circular doubly-linked list using the sibling pointers.

With this representation, we can add or remove nodes from the root list, merge two root lists together, link one two binomial tree to another, or merge a node’s list of children with the root list, in constant time, and we can visit every node in the root list in constant time per node. Having established that these primitive operations can be performed quickly, we never again need to think about the low-level representation details.
L.3 Operations on Fibonacci Heaps

The **Insert**, **Merge**, and **FindMin** algorithms for Fibonacci heaps are exactly like the corresponding algorithms for linked lists. Since we maintain a pointer to the minimum key, **FindMin** is trivial. To insert a new key, we add a single node (which we should think of as a $B_0$) to the root list and (if necessary) update the pointer to the minimum key. To merge two Fibonacci heaps, we just merge the two root lists and keep the pointer to the smaller of the two minimum keys. Clearly, all three operations take $O(1)$ time.

Deleting the minimum key is a little more complicated. First, we remove the minimum key from the root list and splice its children into the root list. Except for updating the parent pointers, this takes $O(1)$ time. Then we scan through the root list to find the new smallest key and update the parent pointers of the new roots. This scan could take $\Theta(n)$ time in the worst case. To bring down the amortized deletion time, we apply a **CleanUp** algorithm, which links pairs of equal-size binomial heaps until there is only one binomial heap of any particular size.

Let me describe the **CleanUp** algorithm in more detail, so we can analyze its running time. The following algorithm maintains a global array $B[1..\lceil \lg n \rceil]$, where $B[i]$ is a pointer to some previously-visited binomial heap of order $i$, or **Null** if there is no such binomial heap. Notice that **CleanUp** simultaneously resets the parent pointers of all the new roots and updates the pointer to the minimum key. I've split off the part of the algorithm that merges binomial heaps of the same order into a separate subroutine **MergeDupes**.

```plaintext
CleanUp:
  newmin ← some node in the root list
  for i ← 0 to \lceil \lg n \rceil
    B[i] ← Null
  for all nodes v in the root list
    parent(v) ← Null  (⋆)
    if key(newmin) > key(v)
      newmin ← v
    MergeDupes(v)

MergeDupes(v):
  w ← B[deg(v)]
  while w ≠ Null
    B[deg(v)] ← Null
    if key(v) ≤ key(w)
      swap v ⇆ w
      remove w from the root list  (⋆⋆)
      link w to v
      w ← B[deg(v)]
    B[deg(v)] ← v
```

![MergeDupes](image)

Notice that **MergeDupes** is careful to merge heaps so that the heap property is maintained—the heap whose root has the larger key becomes a new child of the heap whose root has the smaller key. This is handled by swapping $v$ and $w$ if their keys are in the wrong order.

The running time of **CleanUp** is $O(r')$, where $r'$ is the length of the root list just before **CleanUp** is called. The easiest way to see this is to count the number of times the two starred lines can be executed: line (⋆) is executed once for every node $v$ on the root list, and line (⋆⋆) is executed at most once for every node $w$ on the root list. Since **DeleteMin** does only a constant amount of work before calling **CleanUp**, the running time of **DeleteMin** is $O(r') = O(r + \deg(min))$ where $r$ is the number of roots before **DeleteMin** begins, and min is the node deleted.

---

Please note that the diagram is not included in the text representation. The diagram is a visual representation of the **CleanUp** and **MergeDupes** algorithms, showing the process of merging binomial heaps and demonstrating the effect of changing the root list.
Although \( \text{deg}(\text{min}) \) is at most \( \lg n \), we can still have \( r = \Theta(n) \) (for example, if nothing has been deleted yet), so the worst-case time for a \text{DELETEMIN} is \( \Theta(n) \). After a \text{DELETEMIN}, the root list has length \( O(\log n) \), since all the binomial heaps have unique orders and the largest has order at most \( \lfloor \lg n \rfloor \).

L.4 Amortized Analysis of \text{DELETEMIN}

To bound the amortized cost, observe that each insertion increments \( r \). If we charge a constant ‘cleanup tax’ for each insertion, and use the collected tax to pay for the \text{CLEANUP} algorithm, the unpaid cost of a \text{DELETEMIN} is only \( O(\text{deg}(\text{min})) = O(\log n) \).

More formally, define the potential of the Fibonacci heap to be the number of roots. Recall that the amortized time of an operation can be defined as its actual running time plus the increase in potential, provided the potential is initially zero (it is) and we never have negative potential (we never do). Let \( r \) be the number of roots before a \text{DELETEMIN}, and let \( r'' \) denote the number of roots afterwards. The actual cost of \text{DELETEMIN} is \( r + \text{deg}(\text{min}) \), and the number of roots increases by \( r'' - r \), so the amortized cost is \( r'' + \text{deg}(\text{min}) \). Since \( r'' = O(\log n) \) and the degree of any node is \( O(\log n) \), the amortized cost of \text{DELETEMIN} is \( O(\log n) \).

Each \text{INSERT} adds only one root, so its amortized cost is still constant. A \text{MERGE} actually doesn’t change the number of roots, since the new Fibonacci heap has all the roots from its constituents and no others, so its amortized cost is also constant.

L.5 Decreasing Keys

In some applications of heaps, we also need the ability to delete an arbitrary node. The usual way to do this is to decrease the node’s key to \(-\infty\), and then use \text{DELETEMIN}. Here I’ll describe how to decrease the key of a node in a Fibonacci heap; the algorithm will take \( O(\log n) \) time in the worst case, but the amortized time will be only \( O(1) \).

Our algorithm for decreasing the key at a node \( v \) follows two simple rules.

1. Promote \( v \) up to the root list. (This moves the whole subtree rooted at \( v \).)
2. As soon as two children of any node \( w \) have been promoted, immediately promote \( w \).

In order to enforce the second rule, we now mark certain nodes in the Fibonacci heap. Specifically, a node is marked if exactly one of its children has been promoted. If some child of a marked node is promoted, we promote (and unmark) that node as well. Whenever we promote a marked node, we unmark it; this is the only way to unmark a node. (Specifically, splicing nodes into the root list during a \text{DELETEMIN} is not considered a promotion.)

Here’s a more formal description of the algorithm. The input is a pointer to a node \( v \) and the new value \( k \) for its key.

<table>
<thead>
<tr>
<th><strong>DECREASEKEY</strong>*(( v, k ))**</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{key}(v) \leftarrow k )</td>
</tr>
<tr>
<td>update the pointer to the smallest key</td>
</tr>
<tr>
<td>( \text{PROMOTE}(v) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>PROMOTE</strong>*(( v ))**</th>
</tr>
</thead>
<tbody>
<tr>
<td>unmark ( v )</td>
</tr>
<tr>
<td>if ( \text{parent}(v) \neq \text{NULL} )</td>
</tr>
<tr>
<td>remove ( v ) from ( \text{parent}(v) )'s list of children</td>
</tr>
<tr>
<td>insert ( v ) into the root list</td>
</tr>
<tr>
<td>if ( \text{parent}(v) ) is marked</td>
</tr>
<tr>
<td>( \text{PROMOTE}(\text{parent}(v)) )</td>
</tr>
<tr>
<td>else</td>
</tr>
<tr>
<td>mark ( \text{parent}(v) )</td>
</tr>
</tbody>
</table>
The Promote algorithm calls itself recursively, resulting in a 'cascading promotion'. Each consecutive marked ancestor of \( v \) is promoted to the root list and unmarked, otherwise unchanged. The lowest unmarked ancestor is then marked, since one of its children has been promoted.

Decreasing the keys of four nodes: first \( f \), then \( d \), then \( j \), and finally \( h \). Dark nodes are marked.

\( \text{DecreaseKey}(h) \) causes nodes \( b \) and \( a \) to be recursively promoted.

The time to decrease the key of a node \( v \) is \( O(1 + \#\text{consecutive marked ancestors of } v) \). Binomial heaps have logarithmic depth, so if we still had only full binomial heaps, the running time would be \( O(\log n) \). Unfortunately, promoting nodes destroys the nice binomial tree structure; our trees no longer have logarithmic depth! In fact, **DecreaseKey runs in \( \Theta(n) \) time in the worst case.**

To compute the amortized cost of DecreaseKey, we’ll use the potential method, just as we did for DeleteMin. We need to find a potential function \( \Phi \) that goes up a little whenever we do a little work, and goes down a lot whenever we do a lot of work. DecreaseKey unmarks several marked ancestors and possibly also marks one node. So the number of marked nodes might be an appropriate potential function here. Whenever we do a little bit of work, the number of marks goes up by at most one; whenever we do a lot of work, the number of marks goes down a lot.

More precisely, let \( m \) and \( m' \) be the number of marked nodes before and after a DecreaseKey operation. The actual time (ignoring constant factors) is

\[
t = 1 + \#\text{consecutive marked ancestors of } v
\]

and if we set \( \Phi = m \), the increase in potential is

\[
m' - m \leq 1 - \#\text{consecutive marked ancestors of } v.
\]

Since \( t + \Delta \Phi \leq 2 \), the amortized cost of DecreaseKey is \( O(1) \).

L.6 Bounding the Degree

But now we have a problem with our earlier analysis of DeleteMin. The amortized time for a DeleteMin is still \( O(r + \text{deg(min)}) \). To show that this equaled \( O(\log n) \), we used the fact that the maximum degree of any node is \( O(\log n) \), which implies that after a CleanUp the number of roots is \( O(\log n) \). But now that we don’t have complete binomial heaps, this ‘fact’ is no longer obvious!

So let’s prove it. For any node \( v \), let \(|v|\) denote the number of nodes in the subtree of \( v \), including \( v \) itself. Our proof uses the following lemma, which finally tells us why these things are called Fibonacci heaps.

**Lemma 1.** For any node \( v \) in a Fibonacci heap, \(|v| \geq F_{\text{deg}(v)+2}\).
Proof: Label the children of $v$ in the chronological order in which they were linked to $v$. Consider the situation just before the $i$th oldest child $w_i$ was linked to $v$. At that time, $v$ had at least $i - 1$ children (possibly more). Since CLEANUP only links trees with the same degree, we had $\deg(w_i) = \deg(v) \geq i - 1$. Since that time, at most one child of $w_i$ has been promoted away; otherwise, $w_i$ would have been promoted to the root list by now. So currently we have $\deg(w_i) \geq i - 2$.

We also quickly observe that $\deg(w_i) \geq 0$. (Duh.)

Let $s_d$ be the minimum possible size of a tree with degree $d$ in any Fibonacci heap. Clearly $s_0 = 1$; for notational convenience, let $s_{-1} = 1$ also. By our earlier argument, the $i$th oldest child of the root has degree at least $\max\{0, i - 2\}$, and thus has size at least $\max\{1, s_{i-2}\} = s_{i-2}$. Thus, we have the following recurrence:

$$s_d \geq 1 + \sum_{i=1}^{d} s_{i-2}$$

If we assume inductively that $s_i \geq F_{i+2}$ for all $-1 \leq i < d$ (with the easy base cases $s_{-1} = F_1$ and $s_0 = F_2$), we have

$$s_d \geq 1 + \sum_{i=1}^{d} F_i = F_{d+2}.$$  

(The last step was a practice problem in Homework 0.) By definition, $|v| \geq s_{\deg(v)}$. □

You can easily show (using either induction or the annihilator method) that $F_{k+2} > \phi^k$ where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio. Thus, Lemma 1 implies that

$$\deg(v) \leq \log_\phi |v| = O(\log |v|).$$

Thus, since the size of any subtree in an $n$-node Fibonacci heap is obviously at most $n$, the degree of any node is $O(\log n)$, which is exactly what we wanted. Our earlier analysis is still good.

L.7 Analyzing Everything Together

Unfortunately, our analyses of DELETEMIN and DECREASEKEY used two different potential functions. Unless we can find a single potential function that works for both operations, we can’t claim both amortized time bounds simultaneously. So we need to find a potential function $\Phi$ that goes up a little during a cheap DELETEMIN or a cheap DECREASEKEY, and goes down a lot during an expensive DELETEMIN or an expensive DECREASEKEY.

Let’s look a little more carefully at the cost of each Fibonacci heap operation, and its effect on both the number of roots and the number of marked nodes, the things we used as out earlier potential functions. Let $r$ and $m$ be the numbers of roots and marks before each operation, and let $r'$ and $m'$ be the numbers of roots and marks after the operation.

<table>
<thead>
<tr>
<th>operation</th>
<th>actual cost</th>
<th>$r' - r$</th>
<th>$m' - m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>INSERT</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>MERGE</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>DELETEMIN</td>
<td>$r + \deg(min)$</td>
<td>$r' - r$</td>
<td>0</td>
</tr>
<tr>
<td>DECREASEKEY</td>
<td>$1 + m - m'$</td>
<td>$1 + m - m'$</td>
<td>$m' - m$</td>
</tr>
</tbody>
</table>

In particular, notice that promoting a node in DECREASEKEY requires constant time and increases the number of roots by one, and that we promote (at most) one unmarked node.
If we guess that the correct potential function is a linear combination of our old potential functions $r$ and $m$ and play around with various possibilities for the coefficients, we will eventually stumble across the correct answer:

$$\Phi = r + 2m$$

To see that this potential function gives us good amortized bounds for every Fibonacci heap operation, let’s add two more columns to our table.

<table>
<thead>
<tr>
<th>operation</th>
<th>actual cost</th>
<th>$r' - r$</th>
<th>$m' - m$</th>
<th>$\Phi' - \Phi$</th>
<th>amortized cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>INSERT</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>MERGE</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>DELETE MIN</td>
<td>$r + \deg(\min)$</td>
<td>$r' - r$</td>
<td>0</td>
<td>$r' - r$</td>
<td>$r' + \deg(\min)$</td>
</tr>
<tr>
<td>DECREASE KEY</td>
<td>$1 + m - m'$</td>
<td>$1 + m - m'$</td>
<td>$m' - m$</td>
<td>$1 + m' - m$</td>
<td>2</td>
</tr>
</tbody>
</table>

Since Lemma 1 implies that $r' + \deg(\min) = O(\log n)$, we’re finally done! (Whew!)

### L.8 Fibonacci Trees

To give you a little more intuition about how Fibonacci heaps behave, let’s look at a worst-case construction for Lemma 1. Suppose we want to remove as many nodes as possible from a binomial heap of order $k$, by promoting various nodes to the root list, but without causing any cascading promotions. The most damage we can do is to promote the largest subtree of every node. Call the result a **Fibonacci tree** of order $k + 1$, and denote it $f_{k+1}$. As a base case, let $f_1$ be the tree with one (unmarked) node, that is, $f_1 = B_0$. The reason for shifting the index should be obvious after a few seconds.

Recall that the root of a binomial tree $B_k$ has $k$ children, which are roots of $B_0, B_1, \ldots, B_{k-1}$. To convert $B_k$ to $f_{k+1}$, we promote the root of $B_{k-1}$, and recursively convert each of the other subtrees $B_i$ to $f_{i+1}$. The root of the resulting tree $f_{k+1}$ has degree $k - 1$, and the children are the roots of smaller Fibonacci trees $f_1, f_2, \ldots, f_{k-1}$. We can also consider $B_k$ as two copies of $B_{k-1}$ linked together. It’s quite easy to show that an order-$k$ Fibonacci tree consists of an order $k - 2$ Fibonacci tree linked to an order $k - 1$ Fibonacci tree. (See the picture below.)

Since $f_1$ and $f_2$ both have exactly one node, the number of nodes in an order-$k$ Fibonacci tree is exactly the $k$th Fibonacci number! (That’s why we changed in the index.) Like binomial trees, Fibonacci trees have lots of other nice properties that easy to prove by induction (hint, hint):
The root of $f_k$ has degree $k - 2$.

- $f_k$ can be obtained from $f_{k-1}$ by adding a new unmarked child to every marked node and then marking all the old unmarked nodes.
- $f_k$ has height $\lceil k/2 \rceil - 1$.
- $f_k$ has $F_{k-2}$ unmarked nodes, $F_{k-1}$ marked nodes, and thus $F_k$ nodes altogether.
- $f_k$ has $\binom{k-d-2}{d-1}$ unmarked nodes, $\binom{k-d-2}{d}$ marked nodes, and $\binom{k-d-1}{d}$ total nodes at depth $d$, for all $0 \leq d \leq \lceil k/2 \rceil - 1$.
- $f_k$ has $F_{k-2h-1}$ nodes with height $h$, for all $0 \leq h \leq \lceil k/2 \rceil - 1$, and one node (the root) with height $\lceil k/2 \rceil - 1$.

I stopped covering Fibonacci heaps in my undergraduate algorithms class a few years ago, even though they are a great illustration of data structure design principles and amortized analysis. My main reason for skipping them is that the algorithms are relatively complicated, which both hinders understanding and limits any practical advantages over regular binary heaps. (The popular algorithms textbook CLRS dismisses Fibonacci heaps as “predominantly of theoretical interest” because of programming complexity and large constant factors in its running time.)

Nevertheless, students interested in data structures are strongly advised to become familiar with binomial and Fibonacci heaps, since they share key structural properties with other data structures (such as union-find trees and Bentley-Saxe dynamization) and illustrate important data structure techniques (like lazy rebuilding). Also, Fibonacci heaps (and more recent extensions like Chazelle’s soft heaps) are key ingredients in the fastest algorithms known for some problems.