CS 473G: Combinatorial Algorithms, Fall 2005
Homework 6

Practice only; nothing to turn in.

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1. A small airline, Ivy Air, flies between three cities: Ithaca (a small town in upstate New York), Newark (an eyesore in beautiful New Jersey), and Boston (a yuppie town in Massachusetts). They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Ithaca, stops in Newark, and continues to Boston. There are three types of passengers:

   (a) Those traveling from Ithaca to Newark (god only knows why).

   (b) Those traveling from Newark to Boston (a very good idea).

   (c) Those traveling from Ithaca to Boston (it depends on who you know).

   The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

   (a) Y class: full coach.

   (b) B class: nonrefundable.

   (c) M class: nonrefundable, 3-week advanced purchase.

   Ticket prices, which are largely determined by external influences (i.e., competitors), have been set and advertised as follows:

	Ithaca-Newark	Newark-Boston	Ithaca-Boston
Y	300	160	360
B	220	130	280
M	100	80	140

   Based on past experience, demand forecasters at Ivy Air have determined the following upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations:

	Ithaca-Newark	Newark-Boston	Ithaca-Boston
Y	4	8	3
B	8	13	10
M	22	20	18

   The goal is to decide how many tickets from each of the 9 origin/destination/fare-class combinations to sell. The constraints are that the place cannot be overbooked on either the two legs of the flight and that the number of tickets made available cannot exceed the forecasted maximum demand. The objective is to maximize the revenue.

   Formulate this problem as a linear programming problem.

2. (a) Suppose we are given a directed graph $G = (V, E)$, a length function $\ell : E \to \mathbb{R}$, and a source vertex $s \in V$. Write a linear program to compute the shortest-path distance from s to every other vertex in V. *[Hint: Define a variable for each vertex representing its distance from s. What objective function should you use?]*

   (b) In the *minimum-cost multicommodity-flow* problem, we are given a directed graph $G = (V, E)$, in which each edge $u \to v$ has an associated nonnegative *capacity* $c(u \to v) \geq 0$ and an associated *cost* $\alpha(u \to v)$. We are given k different commodities, each specified by a triple $K_i = (s_i, t_i, d_i)$, where s_i is the source node of the commodity, t_i is the target node for the commodity i, and d_i is the *demand*: the desired flow of commodity i from s_i to t_i. A *flow* for commodity i is a non-negative function $f_i : E \to \mathbb{R}_{\geq 0}$ such that the total flow into any vertex other than s_i or t_i is equal to the total flow out of that vertex. The *aggregate flow* $F : E \to \mathbb{R}$ is defined as the sum of these individual flows: $F(u \to v) = \sum_{i=1}^{k} f_i(u \to v)$. The aggregate flow $F(u \to v)$ on any edge must not exceed the capacity $c(u \to v)$. The goal is to find an aggregate flow whose total *cost* $\sum_{u \to v} F(u \to v) \cdot \alpha(u \to v)$ is as small as possible. (Costs may be negative!) Express this problem as a linear program.

3. In class we described the duality transformation only for linear programs in canonical form:

Primal (Π)		**Dual (∐)**
$\max\ c \cdot x$ s.t. $Ax \leq b$ $x \geq 0$	$\Longleftrightarrow$	$\min\ y \cdot b$ s.t. $yA \geq c$ $y \geq 0$

Describe precisely how to dualize the following more general linear programming problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for each } i = 1 \mathinner{..} p \\ & \sum_{j=1}^{d} a_{ij} x_j = b_i \quad \text{for each } i = p+1 \mathinner{..} p+q \\ & \sum_{j=1}^{d} a_{ij} x_j \geq b_i \quad \text{for each } i = p+q+1 \mathinner{..} n \end{aligned}$$

Your dual problem should have one variable for each primal constraint, and the dual of your dual program should be precisely the original linear program.

4. (a) Model the maximum-cardinality bipartite matching problem as a linear programming problem. The input is a bipartite graph $G = (U, V; E)$, where $E \subseteq U \times V$; the output is the largest matching in G. Your linear program should have one variable for every edge.

   (b) Now dualize the linear program from part (a). What do the dual variables represent? What does the objective function represent? What problem is this!?

5. An *integer program* is a linear program with the additional constraint that the variables must take only integer values.

   (a) Prove that deciding whether an integer program has a feasible solution is NP-complete.

   (b) Prove that finding the optimal feasible solution to an integer program is NP-hard.

   [Hint: Almost any NP=hard decision problem can be rephrased as an integer program. Pick your favorite.]

6. Consider the LP formulation of the shortest path problem presented in class:

$$\begin{array}{lll} \text{maximize} & d_t & \\ \text{subject to} & d_s = 0 & \\ & d_v - d_u \le \ell_{u\to v} & \text{for every edge } u \to v \end{array}$$

   Characterize the feasible bases for this linear program in terms of the original weighted graph. What does a simplex pivoting operation represent? What is a locally optimal (*i.e.*, dual feasible) basis? What does a dual pivoting operation represent?

7. Consider the LP formulation of the maximum-flow problem presented in class:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_w f_{s\to w} - \sum_u f_{u\to s} & \\ \text{subject to} & \displaystyle\sum_w f_{v\to w} - \sum_u f_{u\to v} = 0 & \text{for every vertex } v \ne s,t \\ & f_{u\to v} \le c_{u\to v} & \text{for every edge } u \to v \\ & f_{u\to v} \ge 0 & \text{for every edge } u \to v \end{array}$$

   Is the Ford-Fulkerson augmenting path algorithm an instance of the simplex algorithm applied to this linear program? Why or why not?

*8. *Helly's theorem* says that for any collection of convex bodies in $\mathbb{R}^n$, if every $n+1$ of them intersect, then there is a point lying in the intersection of all of them. Prove Helly's theorem for the special case that the convex bodies are halfspaces. *[Hint: Show that if a system of inequalities $Ax \ge b$ does not have a solution, then we can select $n+1$ of the inequalities such that the resulting system does not have a solution. Construct a primal LP from the system by choosing a 0 cost vector.]*