

You have 180 minutes to answer all seven questions.  
**Write your answers in the separate answer booklet.**  
 You can keep everything except your answer booklet when you leave.

1. An *integer program* is a linear program with the additional constraint that the variables must take only integer values. **Prove** that deciding whether an integer program has a feasible solution is NP-complete. [Hint: *Almost any NP-hard decision problem can be formulated as an integer program. Pick your favorite.*]
2. Recall that a *priority search tree* is a binary tree in which every node has both a *search key* and a *priority*, arranged so that the tree is simultaneously a binary search tree for the keys and a min-heap for the priorities. A *heater* is a priority search tree in which the priorities are given by the user, and the search keys are distributed uniformly and independently at random in the real interval  $[0, 1]$ . Intuitively, a heater is the ‘opposite’ of a treap.

The following problems consider an  $n$ -node heater  $T$  whose node priorities are the integers from 1 to  $n$ . We identify nodes in  $T$  by their priorities; thus, ‘node 5’ means the node in  $T$  with priority 5. The min-heap property implies that node 1 is the root of  $T$ . Finally, let  $i$  and  $j$  be integers with  $1 \leq i < j \leq n$ .

- (a) **Prove** that in a random permutation of the  $(i + 1)$ -element set  $\{1, 2, \dots, i, j\}$ , elements  $i$  and  $j$  are adjacent with probability  $2/(i + 1)$ .
  - (b) **Prove** that node  $i$  is an ancestor of node  $j$  with probability  $2/(i + 1)$ . [Hint: Use part (a)!]
  - (c) What is the probability that node  $i$  is a *descendant* of node  $j$ ? [Hint: **Don’t** use part (a)!]
  - (d) What is the *exact* expected depth of node  $j$ ?
3. The UIUC Faculty Senate has decided to convene a committee to determine whether Chief Illiniwek should become the official ~~mascot~~ *symbol* of the University of Illinois Global Campus. Exactly one faculty member must be chosen from each academic department to serve on this committee. Some faculty members have appointments in multiple departments, but each committee member will represent only one department. For example, if Prof. Blagojevich is affiliated with both the Department of Corruption and the Department of Stupidity, and he is chosen as the Stupidity representative, then someone else must represent Corruption. Finally, University policy requires that any committee on virtual ~~mascots~~ *symbols* must contain the same number of assistant professors, associate professors, and full professors. Fortunately, the number of departments is a multiple of 3.  
 Describe an efficient algorithm to select the membership of the Global Illiniwek Committee. Your input is a list of all UIUC faculty members, their ranks (assistant, associate, or full), and their departmental affiliation(s). There are  $n$  faculty members and  $3k$  departments.
  4. Let  $\alpha(G)$  denote the number of vertices in the largest independent set in a graph  $G$ . **Prove** that the following problem is NP-hard: Given a graph  $G$ , return *any* integer between  $\alpha(G) - 31337$  and  $\alpha(G) + 31337$ .

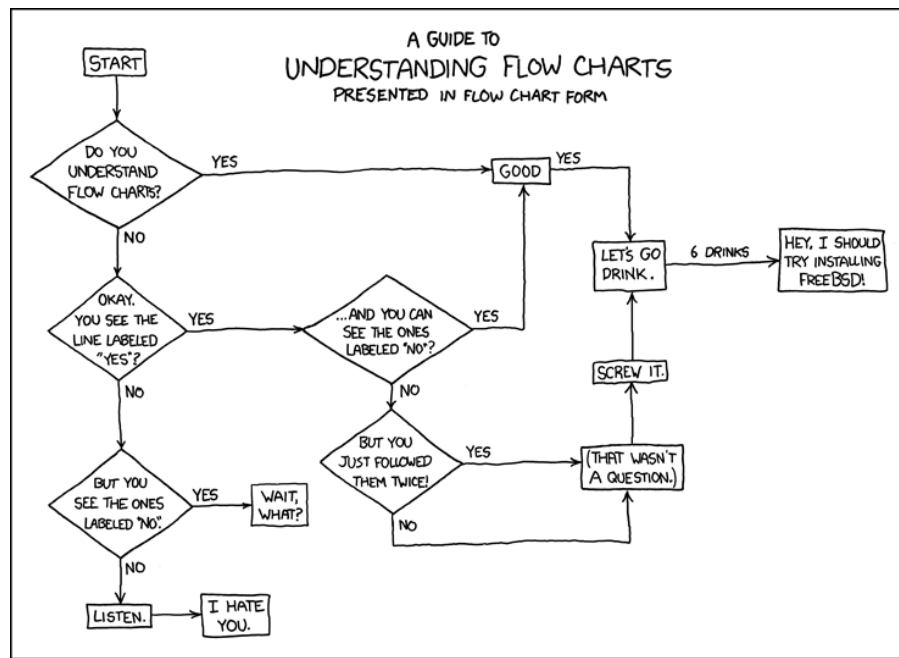
5. Let  $G = (V, E)$  be a directed graph with capacities  $c : E \rightarrow \mathbb{R}^+$ , a source vertex  $s$ , and a target vertex  $t$ . Suppose someone hands you a function  $f : E \rightarrow \mathbb{R}$ . Describe and analyze a fast algorithm to determine whether  $f$  is a maximum  $(s, t)$ -flow in  $G$ .

6. For some strange reason, you decide to ride your bicycle 3688 miles from Urbana to Wasilla, Alaska, to join in the annual Wasilla Mining Festival and Helicopter Wolf Hunt. The festival starts exactly 32 days from now, so you need to bike an average of 109 miles each day. Because you are a poor starving student, you can only afford to sleep at campgrounds, which are unfortunately *not* spaced exactly 109 miles apart. So some days you will have to ride more than average, and other days less, but you would like to keep the variation as small as possible. You settle on a formal scoring system to help decide where to sleep; if you ride  $x$  miles in one day, your score for that day is  $(109 - x)^2$ . What is the minimum possible total score for all 32 days?

More generally, suppose you have  $D$  days to travel  $DP$  miles, there are  $n$  campgrounds along your route, and your score for traveling  $x$  miles in one day is  $(x - P)^2$ . You are given a sorted array  $dist[1..n]$  of real numbers, where  $dist[i]$  is the distance from your starting location to the  $i$ th campground; it may help to also set  $dist[0] = 0$  and  $dist[n + 1] = DP$ . Describe and analyze a fast algorithm to compute the minimum possible score for your trip. The running time of your algorithm should depend on the integers  $D$  and  $n$ , but not on the real number  $P$ .

7. Describe and analyze efficient algorithms for the following problems.

- (a) Given a set of  $n$  integers, does it contain elements  $a$  and  $b$  such that  $a + b = 0$ ?
- (b) Given a set of  $n$  integers, does it contain elements  $a$ ,  $b$ , and  $c$  such that  $a + b = c$ ?



— Randall Munroe, *xkcd*, December 17, 2008 (<http://xkcd.com/518/>)