CS 573: Graduate Algorithms, Fall 2008
Homework 3

Due at 11:59:59pm, Wednesday, October 22, 2008

- Groups of up to three students may submit a single, common solution. Please neatly print (or typeset) the full name, NetID, and the HW0 alias (if any) of every group member on the first page of your submission.

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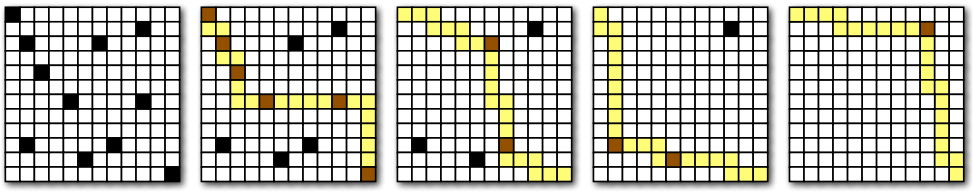
1. Consider an $n \times n$ grid, some of whose cells are marked. A *monotone* path through the grid starts at the top-left cell, moves only right or down at each step, and ends at the bottom-right cell. We want to compute the minimum number of monotone paths that cover all marked cells. The input to our problem is an array $M[1..n, 1..n]$ of booleans, where $M[i, j] = \textsc{True}$ if and only if cell (i, j) is marked.

   One of your friends suggests the following greedy strategy:

   - Find (somehow) one "good" path π that covers the maximum number of marked cells.
   - Unmark the cells covered by π.
   - If any cells are still marked, recursively cover them.

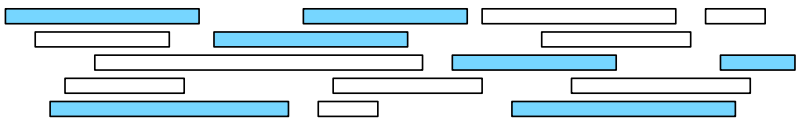
   Does this greedy strategy always compute an optimal solution? If yes, give a proof. If no, give a counterexample.



Greedily covering the marked cells in a grid with four monotone paths.

2. Let X be a set of n intervals on the real line. A subset of intervals $Y \subseteq X$ is called a ***tiling path*** if the intervals in Y cover the intervals in X, that is, any real value that is contained in some interval in X is also contained in some interval in Y. The *size* of a tiling path is just the number of intervals.

   Describe and analyze an algorithm to compute the smallest tiling path of X as quickly as possible. Assume that your input consists of two arrays $X_L[1..n]$ and $X_R[1..n]$, representing the left and right endpoints of the intervals in X. If you use a greedy algorithm, you must prove that it is correct.



A set of intervals. The seven shaded intervals form a tiling path.

3. Given a graph G with edge weights and an integer k, suppose we wish to partition the vertices of G into k subsets $S_1, S_2, \ldots, S_k$ so that the sum of the weights of the edges that cross the partition (*i.e.*, that have endpoints in different subsets) is as large as possible.

   (a) Describe an efficient $(1 - 1/k)$-approximation algorithm for this problem. *[Hint: Solve the special case $k = 2$ first.]*

   (b) Now suppose we wish to minimize the sum of the weights of edges that do *not* cross the partition. What approximation ratio does your algorithm from part (a) achieve for this new problem? Justify your answer.

4. Consider the following heuristic for constructing a vertex cover of a connected graph G: ***Return the set of all non-leaf nodes of any depth-first spanning tree.*** (Recall that a depth-first spanning tree is a *rooted* tree; the root is not considered a leaf, even if it has only one neighbor in the tree.)

   (a) Prove that this heuristic returns a vertex cover of G.

   (b) Prove that this heuristic returns a 2-approximation to the minimum vertex cover of G.

   (c) Prove that for any $\varepsilon > 0$, there is a graph for which this heuristic returns a vertex cover of size at least $(2 - \varepsilon) \cdot OPT$.

5. Consider the following greedy approximation algorithm to find a vertex cover in a graph:

```
GreedyVertexCover(G):
  C ← ∅
  while G has at least one edge
      v ← vertex in G with maximum degree
      G ← G \ v
      C ← C ∪ v
  return C
```

   In class we proved that the approximation ratio of this algorithm is $O(\log n)$; your task is to prove a matching lower bound. Specifically, for any positive integer n, describe an n-vertex graph G such that GreedyVertexCover(G) returns a vertex cover that is $\Omega(\log n)$ times larger than optimal. *[Hint: $H_n = \Omega(\log n)$.]*

*6. ***[Extra credit]*** Consider the greedy algorithm for metric TSP: Start at an arbitrary vertex u, and at each step, travel to the closest unvisited vertex.

   (a) Prove that this greedy algorithm is an $O(\log n)$-approximation algorithm, where n is the number of vertices. *[Hint: Show that the kth least expensive edge in the tour output by the greedy algorithm has weight at most $OPT/(n - k + 1)$; try $k = 1$ and $k = 2$ first.]*

   *(b) Prove that the greedy algorithm for metric TSP is no better than an $O(\log n)$-approximation. That is, describe an infinite family of weighted graphs that satisfy the triangle inequality, such that the greedy algorithm returns a cycle whose length is $\Omega(\log n)$ times the optimal TSP tour.