This exam lasts 120 minutes.
Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

1. Each of these ten questions has one of the following five answers:

   A: $\Theta(1)$ B: $\Theta(\log n)$ C: $\Theta(n)$ D: $\Theta(n \log n)$ E: $\Theta(n^2)$

   (a) What is $\dfrac{n^3 + 3n^2 - 5n + 1}{4n^2 - 2n + \sqrt{3}}$?

   (b) What is $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{i} 5$?

   (c) What is $\displaystyle\sum_{i=1}^{n} \left(\frac{i}{n} + \frac{n}{i}\right)$?

   (d) How many bits are required to write the nth Fibonacci number F_n in binary?

   (e) What is the solution to the recurrence $E(n) = E(n-3) + 2n - 1$?

   (f) What is the solution to the recurrence $F(n) = 2F(n/3) + 2F(n/6) + n$?

   (g) What is the solution to the recurrence $G(n) = 12\,G(n/4) + n^2$?

   (h) What is the worst-case depth of an n-node binary tree?

   (i) Consider the following recursive function, which is defined in terms of a fixed array $X[1\,..\,n]$:

   $$WTF(i,j) = \begin{cases} 0 & \text{if } i \le 0 \text{ or } j \le 0 \\ X[j] + WTF(i-1, j) + WTF(i, \lfloor j/2 \rfloor) & \text{otherwise} \end{cases}$$

   How long does it take to compute $WTF(n,n)$ using dynamic programming?

   (j) How many seconds does it take for a 10-megapixel image taken by the Curiosity Rover to be encoded, transmitted from Mars to Earth, decoded, and tweeted?

2. A palindrome is any string that is exactly the same as its reversal, like I, or DEED, or HANNAH, or SATORAREPOTENETOPERAROTAS. Describe and analyze an algorithm to find the length of the longest subsequence of a given string that is also a palindrome.

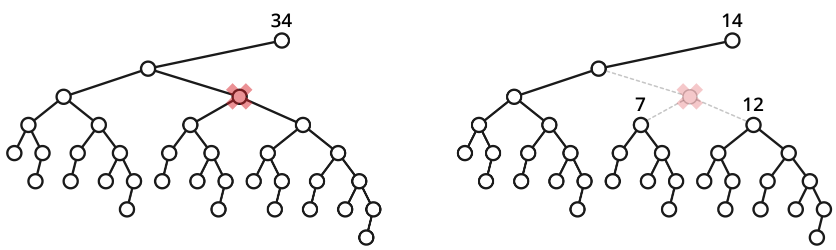
   For example, the longest palindrome subsequence of **M**A**H**D**Y**NA**M**ICP**RO**G**R**AMZLET**M**ESHOW**Y**OUT**H**E**M** is MHYMRORMYHM, so given that string as input, your algorithm should return the integer 11.

3. ***Prove*** that any integer—positive, negative, or zero—can be represented as the sum of distinct powers of -2. For example:

$$
\begin{aligned}
4 &= 4 &&= (-2)^2 \\
8 &= 16-8 &&= (-2)^4+(-2)^3 \\
15 &= 16-2+1 &&= (-2)^4+(-2)^1+(-2)^0 \\
-16 &= -32+16 &&= (-2)^5+(-2)^4 \\
23 &= 64-32-8-2+1 &&= (-2)^6+(-2)^5+(-2)^3+(-2)^1+(-2)^0 \\
-42 &= -32-8-2 &&= (-2)^5+(-2)^3+(-2)^1
\end{aligned}
$$

4. Let T be a binary tree with n vertices. Deleting any vertex v splits T into at most three subtrees, containing the left child of v (if any), the right child of v (if any), and the parent of v (if any). We call v a ***central*** vertex if each of these smaller trees has at most $n/2$ vertices.

   Describe and analyze an algorithm to find a central vertex in a given binary tree.



Deleting a central vertex in a 34-node binary tree,
leaving subtrees with 14 nodes, 7 nodes, and 12 nodes.

5. Let $n = 2^\ell - 1$ for some positive integer ℓ. Suppose someone claims to hold an array $A[1\,..\,n]$ of *distinct* ℓ-bit strings; thus, exactly one ℓ-bit string does *not* appear in A. Suppose further that the ***only*** way we can access A is by calling the function FetchBit(i, j), which returns the jth bit of the string $A[i]$ in $O(1)$ time.

   Describe an algorithm to find the missing string in A using only $O(n)$ calls to FetchBit.