

This exam lasts 120 minutes.

Write your answers in the separate answer booklet.

Please return this question sheet and your cheat sheet with your answers.

1. Each of these ten questions has one of the following five answers:

A: $\Theta(1)$ B: $\Theta(\log n)$ C: $\Theta(n)$ D: $\Theta(n \log n)$ E: $\Theta(n^2)$

(a) What is $\frac{n^5 - 3n^3 - 5n + 4}{4n^3 - 2n^2 + n - 7}$?

(b) What is $\sum_{i=1}^n i$?

(c) What is $\sum_{i=1}^n \sqrt{\frac{n}{i}}$?

(d) How many bits are required to write the integer n^{10} in binary?

(e) What is the solution to the recurrence $E(n) = E(n/2) + E(n/4) + E(n/8) + 16n$?

(f) What is the solution to the recurrence $F(n) = 6F(n/6) + 6n$?

(g) What is the solution to the recurrence $G(n) = 9G(\lceil n/3 \rceil + 1) + n$?

(h) The *total path length* of a binary tree is the sum of the depths of all nodes. What is the total path length of an n -node binary tree in the worst case?

(i) Consider the following recursive function, defined in terms of a fixed array $X[1..n]$:

$$WTF(i, j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \max \left\{ \begin{array}{l} 2 \cdot [X[i] \neq X[j]] + WTF(i+1, j-1) \\ 1 + WTF(i+1, j) \\ 1 + WTF(i, j-1) \end{array} \right\} & \text{otherwise} \end{cases}$$

How long does it take to compute $WTF(1, n)$ using dynamic programming?

(j) Voyager 1 recently became the first made-made object to reach interstellar space. Currently the spacecraft is about 18 billion kilometers (roughly 60,000 light seconds) from Earth, traveling outward at approximately 17 kilometers per second (approximately 1/18000 of the speed of light). Voyager carries a golden record containing over 100 digital images and approximately one hour of sound recordings. In digital form, the recording would require about 1 gigabyte. Voyager can transmit data back to Earth at approximately 1400 bits per second. Suppose the engineers at JPL sent instructions to Voyager 1 to send the complete contents of the Golden Record back to Earth; how many seconds would they have to wait to receive the entire record?

2. You are a visitor at a political convention (or perhaps a faculty meeting) with n delegates; each delegate is a member of exactly one political party. It is impossible to tell which political party any delegate belongs to; in particular, you will be summarily ejected from the convention if you ask. However, you can determine whether any pair of delegates belong to the *same* party or not simply by introducing them to each other—members of the same party always greet each other with smiles and friendly handshakes; members of different parties always greet each other with angry stares and insults.

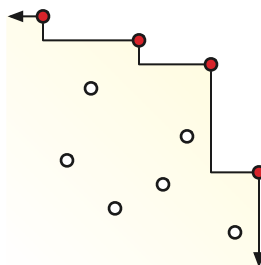
Suppose more than half of the delegates belong to the same political party. Describe and analyze an efficient algorithm that identifies all members of this majority party.

3. Recall that a *tree* is a connected undirected graph with no cycles. **Prove** that in any tree, the number of nodes is exactly one more than the number of edges.
4. Next spring break, you and your friends decide to take a road trip, but before you leave, you decide to figure out *exactly* how much money to bring for gasoline. Suppose you compile a list of all gas stations along your planned route, containing the following information:
- A sorted array $Dist[0..n]$, where $Dist[0] = 0$ and $Dist[i]$ is the number of miles from the beginning of your route to the i th gas station. Your route ends at the n th gas station.
 - A second array $Price[1..n]$, where $Price[i]$ is the price of one gallon of gasoline at the i th gas station. (Unlike in real life, these prices do not change over time.)

You start the trip with a full tank of gas. Whenever you buy gas, you must completely fill your tank. Your car holds exactly 10 gallons of gas and travels exactly 25 miles per gallon; thus, starting with a full tank, you can travel exactly 250 miles before your car dies. Finally, $Dist[i+1] < Dist[i] + 250$ for every index i , so the trip is possible.

Describe and analyze an algorithm to determine the minimum amount of money you must spend on gasoline to guarantee that you can drive the entire route.

5. Suppose you are given a set P of n points in the plane. A point $p \in P$ is *maximal* in P if no other point in P is both above and to the right of p . Intuitively, the maximal points define a “staircase” with all the other points of P below it.



A set of ten points, four of which are maximal.

Describe and analyze an algorithm to compute the number of maximal points in P in $O(n \log n)$ time. For example, given the ten points shown above, your algorithm should return the integer 4.