Let L be an arbitrary regular language.

1. Prove that the language $insert\mathtt{1}(L) := \{x\mathtt{1}y \mid xy \in L\}$ is regular.

   Intuitively, $insert\mathtt{1}(L)$ is the set of all strings that can be obtained from strings in L by inserting exactly one $\mathtt{1}$. For example, if $L = \{\varepsilon, \mathtt{OOK!}\}$, then $insert\mathtt{1}(L) = \{\mathtt{1}, \mathtt{1OOK!}, \mathtt{O1OK!}, \mathtt{OO1K!}, \mathtt{OOK1!}, \mathtt{OOK!1}\}$.

2. Prove that the language $delete\mathtt{1}(L) := \{xy \mid x\mathtt{1}y \in L\}$ is regular.

   Intuitively, $delete\mathtt{1}(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one $\mathtt{1}$. For example, if $L = \{\mathtt{101101}, \mathtt{00}, \varepsilon\}$, then $delete\mathtt{1}(L) = \{\mathtt{01101}, \mathtt{10101}, \mathtt{10110}\}$.

Work on these later: (In fact, these might be easier than problems 1 and 2.)

3. Consider the following recursively defined function on strings:

$$stutter(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \bullet stutter(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

   Intuitively, $stutter(w)$ doubles every symbol in w. For example:

   - $stutter(\mathtt{PRESTO}) = \mathtt{PPRREESSTTOO}$
   - $stutter(\mathtt{HOCUS\diamond POCUS}) = \mathtt{HHOOCCUUSS\diamond\diamond PPOOCCUUSS}$

   Let L be an arbitrary regular language.

   (a) Prove that the language $stutter^{-1}(L) := \{w \mid stutter(w) \in L\}$ is regular.

   (b) Prove that the language $stutter(L) := \{stutter(w) \mid w \in L\}$ is regular.

4. Consider the following recursively defined function on strings:

$$evens(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \\ b \cdot evens(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x \end{cases}$$

   Intuitively, $evens(w)$ skips over every other symbol in w. For example:

   - $evens(\mathtt{EXPELLIARMUS}) = \mathtt{XELAMS}$
   - $evens(\mathtt{AVADA\diamond KEDAVRA}) = \mathtt{VD\diamond EAR}$.

   Once again, let L be an arbitrary regular language.

   (a) Prove that the language $evens^{-1}(L) := \{w \mid evens(w) \in L\}$ is regular.

   (b) Prove that the language $evens(L) := \{evens(w) \mid w \in L\}$ is regular.