CS/ECE 374 A ✧ Fall 2019

Homework 10

Due Tuesday, December 3, 2019 at 8pm

This is the last graded homework before the final exam.

This brings the total number of graded homework problems to 33,
at most 24 of which will count toward your final course grade.

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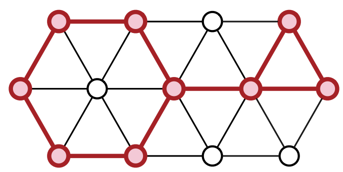
1. A subset S of vertices in an undirected graph G is called ***square-free*** if, for every four distinct vertices $u, v, w, x \in S$, at least one of the four edges uv, vw, wx, xu is *absent* from G. That is, the subgraph of G induced by S has no cycles of length 4. Prove that finding the size of the largest square-free subset of vertices in a given undirected graph is NP-hard.



A square-free subset of 9 vertices, and all edges between them.
This is ***not*** the largest square-free subset in this graph.

2. Fix an alphabet $\Sigma = \{0, 1\}$. Prove that the following problems are NP-hard.[1]

   (a) Given a regular expression R over the alphabet Σ, is $L(R) \neq \Sigma^*$?

   (b) Given an NFA M over the alphabet Σ, is $L(M) \neq \Sigma^*$?

   *[Hint: Encode all the **bad** choices for some problem into a regular expression R, so that if **all** choices are bad, then $L(R) = \Sigma^*$.]*

3. ***This problem has been removed.*** — *We are deferring all discussion of undecidability until after Thanksgiving break. This problem will reappear on "Homework 11".*

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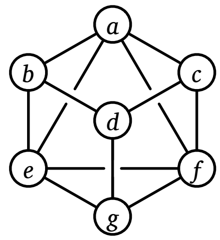
[1] In fact, both of these problems are NP-hard even when $|\Sigma| = 1$, but proving that is much more difficult.

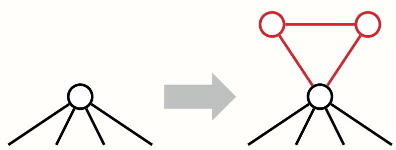
Solved Problem

4. A *double-Hamiltonian tour* in an undirected graph G is a closed walk that visits every vertex in G exactly twice. Prove that it is NP-hard to decide whether a given graph G has a double-Hamiltonian tour.



This graph contains the double-Hamiltonian tour $a \to b \to d \to g \to e \to b \to d \to c \to f \to a \to c \to f \to g \to e \to a$.

Solution: We prove the problem is NP-hard with a reduction from the standard Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a small gadget to every vertex of G. Specifically, for each vertex v, we add two vertices $v^\sharp$ and $v^\flat$, along with three edges $vv^\flat$, $vv^\sharp$, and $v^\flat v^\sharp$.



A vertex in G, and the corresponding vertex gadget in H.

I claim that G has a Hamiltonian cycle if and only if H has a double-Hamiltonian tour.

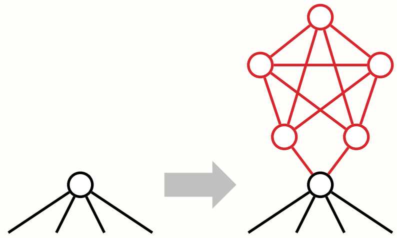
$\Longrightarrow$ Suppose G has a Hamiltonian cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. We can construct a double-Hamiltonian tour of H by replacing each vertex v_i with the following walk:

$$\cdots \to v_i \to v_i^\flat \to v_i^\sharp \to v_i^\flat \to v_i^\sharp \to v_i \to \cdots$$

$\Longleftarrow$ Conversely, suppose H has a double-Hamiltonian tour D. Consider any vertex v in the original graph G; the tour D must visit v exactly twice. Those two visits split D into two closed walks, each of which visits v exactly once. Any walk from $v^\flat$ or $v^\sharp$ to any other vertex in H must pass through v. Thus, one of the two closed walks visits only the vertices v, $v^\flat$, and $v^\sharp$. Thus, if we simply remove the vertices in $H \setminus G$ from D, we obtain a closed walk in G that visits every vertex in G once.

Given any graph G, we can clearly construct the corresponding graph H in polynomial time.

With more effort, we can construct a graph H that contains a double-Hamiltonian tour ***that traverses each edge of H at most once*** if and only if G contains a Hamiltonian cycle. For each vertex v in G we attach a more complex gadget containing five vertices and eleven edges, as shown on the next page.



A vertex in G, and the corresponding modified vertex gadget in H.

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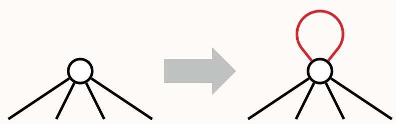
Rubric: 10 points, standard polynomial-time reduction rubric. This is not the only correct solution.

Non-solution (self-loops): We attempt to prove the problem is NP-hard with a reduction from the Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a self-loop every vertex of G. Given any graph G, we can clearly construct the corresponding graph H in polynomial time.

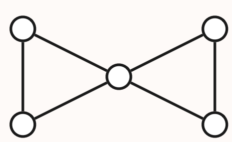


An incorrect vertex gadget.

Suppose G has a Hamiltonian cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. We can construct a double-Hamiltonian tour of H by alternating between edges of the Hamiltonian cycle and self-loops:

$$v_1 \to v_1 \to v_2 \to v_2 \to v_3 \to \cdots \to v_n \to v_n \to v_1.$$

Unfortunately, if H has a double-Hamiltonian tour, we *cannot* conclude that G has a Hamiltonian cycle, because we cannot guarantee that a double-Hamiltonian tour in H uses *any* self-loops. The graph G shown below is a counterexample; it has a double-Hamiltonian tour (even before adding self-loops!) but no Hamiltonian cycle.



This graph has a double-Hamiltonian tour.

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