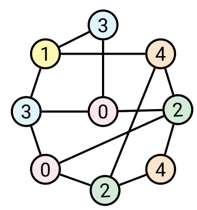
Recall that a proper k-coloring of a graph G is a function that assigns each vertex of G a "color" from the set $\{0, 1, 2, \ldots, k-1\}$ (or less formally, from any set of size k), such that for any edge uv, vertices u and v are assigned different "colors". The *chromatic number* of G is the smallest integer k such that G has a proper k-coloring.

1. A proper k-coloring of a graph G is ***balanced*** if each color is assigned to exactly the same number of vertices. Prove that it is NP-hard to decide whether a given graph G has a *balanced* 3-coloring. *[Hint: Reduce from the standard 3Color problem.]*

2. Prove that the following problem is NP-hard: Given an undirected graph G, find *any* integer $k > 374$ such that G has a proper coloring with k colors but G does not have a proper coloring with $k - 374$ colors. For example, if the chromatic number of G is 10000, then any integer between 10000 and 10373 is a correct answer.

3. A 5-coloring is ***careful*** if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard. *[Hint: Reduce from the standard 5Color problem.]*

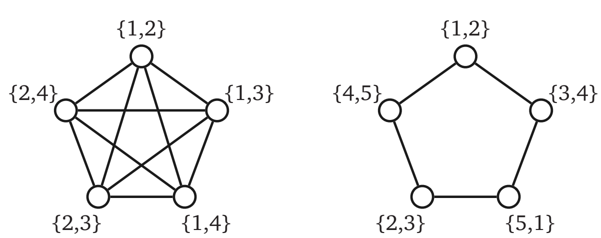


A careful 5-coloring.

4. A ***bicoloring*** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

   (a) Prove that it is NP-hard to determine whether a given graph has a weak bicoloring with three colors. *[Hint: Reduce from the standard 3Color problem.]*

   (b) Prove that it is NP-hard to determine whether a given graph has a strong bicoloring with ***five*** colors. *[Hint: Reduce from the standard 3Color (sic) problem!]*



Left: A weak bicoloring of a 5-clique with four colors.
Right A strong bicoloring of a 5-cycle with five colors.