CS 373: Combinatorial Algorithms, Spring 2001
Homework 3 (due Thursday, March 8, 2001 at 11:59.99 p.m.)

Name:		
Net ID:	Alias:	U $^3/_4$ 1

Name:		
Net ID:	Alias:	U $^3/_4$ 1

Name:		
Net ID:	Alias:	U $^3/_4$ 1

Starting with Homework 1, homeworks may be done in teams of up to three people. Each team turns in just one solution, and every member of a team gets the same grade. Since 1-unit graduate students are required to solve problems that are worth extra credit for other students, **1-unit grad students may not be on the same team as 3/4-unit grad students or undergraduates.**

Neatly print your name(s), NetID(s), and the alias(es) you used for Homework 0 in the boxes above. Please also tell us whether you are an undergraduate, 3/4-unit grad student, or 1-unit grad student by circling U, $^3/_4$, or 1, respectively. Staple this sheet to the top of your homework.

Required Problems

1. Hashing:
   A hash table of size m is used to store n items with $n \leq m/2$. Open addressing is used for collision resolution.

   (a) Assuming uniform hashing, show that for $i = 1, 2, \ldots, n$, the probability that the i^{th} insertion requires strictly more than k probes is at most 2^{-k}.

   (b) Show that for $i = 1, 2, \ldots, n$, the probability that the i^{th} insertion requires more than $2 \lg n$ probes is at most $1/n^2$.

   Let the random variable X_i denote the number of probes required by the i^{th} insertion. You have shown in part (b) that $\Pr\{X_i > 2 \lg n\} \leq 1/n^2$. Let the random variable $X = \max_{1 \leq i \leq n} X_i$ denote the maximum number of probes required by any of the n insertions.

   (c) Show that $\Pr\{X > 2 \lg n\} \leq 1/n$.

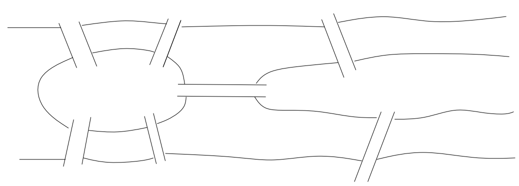
   (d) Show that the expected length of the longest probe sequence is $E[X] = O(\lg n)$.

2. Reliable Network:
Suppose you are given a graph of a computer network $G = (V, E)$ and a function $r(u, v)$ that gives a reliability value for every edge $(u, v) \in E$ such that $0 \leq r(u, v) \leq 1$. The reliability value gives the probability that the network connection corresponding to that edge will *not* fail. Describe and analyze an algorithm to find the most reliable path from a given source vertex s to a given target vertex t.

3. Aerophobia:
After graduating you find a job with Aerophobes-R′-Us, the leading traveling agency for aerophobic people. Your job is to build a system to help customers plan airplane trips from one city to another. All of your customers are afraid of flying so the trip should be as short as possible.

   In other words, a person wants to fly from city A to city B in the shortest possible time. S/he turns to the traveling agent who knows all the departure and arrival times of all the flights on the planet. Give an algorithm that will allow the agent to choose an optimal route to minimize the total time in transit. Hint: rather than modify Dijkstra's algorithm, modify the data. The total transit time is from departure to arrival at the destination, so it will include layover time (time waiting for a connecting flight).

4. The Seven Bridges of Königsberg:
During the eighteenth century the city of Königsberg in East Prussia was divided into four sections by the Pregel river. Seven bridges connected these regions, as shown below. It was said that residents spent their Sunday walks trying to find a way to walk about the city so as to cross each bridge exactly once and then return to their starting point.



   (a) Show how the residents of the city could accomplish such a walk or prove no such walk exists.

   (b) Given any undirected graph $G = (V, E)$, give an algorithm that finds a cycle in the graph that visits every edge exactly once, or says that it can't be done.

5. Minimum Spanning Tree changes:
Suppose you have a graph G and an MST of that graph (i.e. the MST has already been constructed).

   (a) Give an algorithm to update the MST when an edge is added to G.

   (b) Give an algorithm to update the MST when an edge is deleted from G.

   (c) Give an algorithm to update the MST when a vertex (and possibly edges to it) is added to G.

6. Nesting Envelopes
[This problem is required only for graduate students taking CS 373 for a full unit; anyone else can submit a solution for extra credit.] You are given an unlimited number of each of n different types of envelopes. The dimensions of envelope type i are $x_i \times y_i$. In nesting envelopes inside one another, you can place envelope A inside envelope B if and only if the dimensions A are *strictly smaller* than the dimensions of B. Design and analyze an algorithm to determine the largest number of envelopes that can be nested inside one another.

Practice Problems

1. Makefiles:
In order to facilitate recompiling programs from multiple source files when only a small number of files have been updated, there is a UNIX utility called 'make' that only recompiles those files that were changed after the most recent compilation, *and* any intermediate files in the compilation that depend on those that were changed. A Makefile is typically composed of a list of source files that must be compiled. Each of these source files is dependent on some of the other files which are listed. Thus a source file must be recompiled if a file on which it depends is changed.

   Assuming you have a list of which files have been recently changed, as well as a list for each source file of the files on which it depends, design an algorithm to recompile only those necessary. DO NOT worry about the details of parsing a Makefile.

★2. Let the hash function for a table of size m be

$$h(x) = \lfloor Amx \rfloor \bmod m$$

   where $A = \hat{\phi} = \frac{\sqrt{5}-1}{2}$. Show that this gives the best possible spread, i.e. if the x are hashed in order, $x + 1$ will be hashed in the largest remaining contiguous interval.

3. The incidence matrix of an undirected graph $G = (V, E)$ is a $|V| \times |E|$ matrix $B = (b_{ij})$ such that

$$b_{ij} = \begin{cases} 1 & (i, j) \in E, \\ 0 & (i, j) \notin E. \end{cases}$$

   (a) Describe what all the entries of the matrix product BB^T represent (B^T is the matrix transpose). Justify.

   (b) Describe what all the entries of the matrix product $B^T B$ represent. Justify.

   ★(c) Let $C = BB^T - 2A$. Let C' be C with the first row and column removed. Show that $\det C'$ is the number of spanning trees. (A is the adjacency matrix of G, with zeroes on the diagonal).

4. $o(V^2)$ Adjacency Matrix Algorithms

   (a) Give an $O(V)$ algorithm to decide whether a directed graph contains a *sink* in an adjacency matrix representation. A sink is a vertex with in-degree $V - 1$.

(b) An undirected graph is a scorpion if it has a vertex of degree 1 (the sting) connected to a vertex of degree two (the tail) connected to a vertex of degree $V-2$ (the body) connected to the other $V-3$ vertices (the feet). Some of the feet may be connected to other feet.

Design an algorithm that decides whether a given adjacency matrix represents a scorpion by examining only $O(V)$ of the entries.

(c) Show that it is impossible to decide whether G has at least one edge in $O(V)$ time.

5. Shortest Cycle:
Given an **undirected** graph $G = (V, E)$, and a weight function $f : E \to \mathbf{R}$ on the **edges**, give an algorithm that finds (in time polynomial in V and E) a cycle of smallest weight in G.

6. Longest Simple Path:
Let graph $G = (V, E)$, $|V| = n$. A *simple path* of G, is a path that does not contain the same vertex twice. Use dynamic programming to design an algorithm (not polynomial time) to find a simple path of maximum length in G. Hint: It can be done in $O(n^c 2^n)$ time, for some constant c.

7. Minimum Spanning Tree:
Suppose all edge weights in a graph G are equal. Give an algorithm to compute an MST.

8. Transitive reduction:
Give an algorithm to construct a *transitive reduction* of a directed graph G, i.e. a graph G^{TR} with the fewest edges (but with the same vertices) such that there is a path from a to b in G iff there is also such a path in G^{TR}.

9. (a) What is $5^{2^{29^{5^0}+23^{4^1}+17^{3^2}+11^{2^3}+5^{1^4}}} \bmod 6$?

(b) What is the capital of Nebraska? Hint: It is not Omaha. It is named after a famous president of the United States that was not George Washington. The distance from the Earth to the Moon averages roughly 384,000 km.