

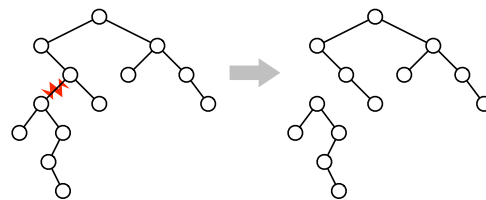
You have 180 minutes to answer six of the seven questions.  
Write your answers in the separate answer booklet.  
You may take the question sheet with you when you leave.

1. SUBSETSUM and PARTITION are two closely related NP-hard problems, defined as follows.

**SUBSETSUM:** Given a set  $X$  of positive integers and a positive integer  $k$ , does  $X$  have a subset whose elements sum up to  $k$ ?

**PARTITION:** Given a set  $Y$  of positive integers, can  $Y$  be partitioned into two subsets whose sums are equal?

- (a) [2 pts] *Prove* that PARTITION and SUBSETSUM are both in NP.
- (b) [1 pt] Suppose you already know that SUBSETSUM is NP-hard. Which of the following arguments could you use to prove that PARTITION is NP-hard? *You do not need to justify your answer* — just answer ① or ②.
- ① Given a set  $X$  and an integer  $k$ , construct a set  $Y$  in polynomial time, such that PARTITION( $Y$ ) is true if and only if SUBSETSUM( $X, k$ ) is true.
- ② Given a set  $Y$ , construct a set  $X$  and an integer  $k$  in polynomial time, such that PARTITION( $Y$ ) is true if and only if SUBSETSUM( $X, k$ ) is true.
- (c) [3 pts] Describe and analyze a polynomial-time reduction from PARTITION to SUBSETSUM. *You do not need to prove that your reduction is correct.*
- (d) [4 pts] Describe and analyze a polynomial-time reduction from SUBSETSUM to PARTITION. *You do not need to prove that your reduction is correct.*
2. (a) [4 pts] For any node  $v$  in a binary tree, let  $size(v)$  denote the number of nodes in the subtree rooted at  $v$ . Let  $k$  be an arbitrary positive number. *Prove* that every binary tree with at least  $k$  nodes contains a node  $v$  such that  $k \leq size(v) \leq 2k$ .
- (b) [2 pts] Removing any edge from an  $n$ -node binary tree  $T$  separates it into two smaller binary trees. An edge is called a **balanced separator** if both of these subtrees have at least  $n/3$  nodes (and therefore at most  $2n/3$  nodes). *Prove* that every binary tree with more than one node has a balanced separator. [Hint: Use part (a).]
- (c) [4 pts] Describe and analyze an algorithm to find a balanced separator in a given binary tree. [Hint: Use part (a).]



Removing a balanced separator from a binary tree.

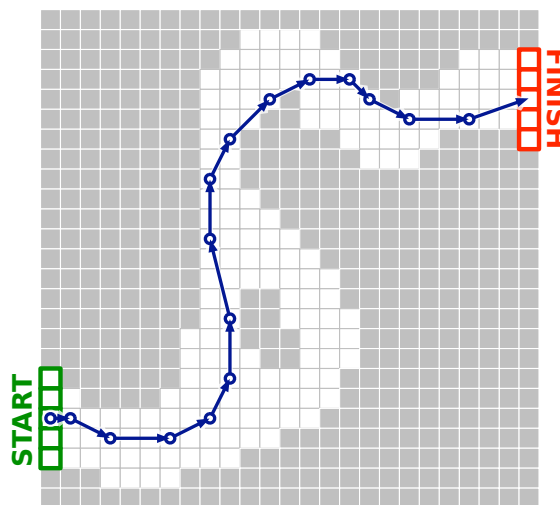
3. **Racetrack** (also known as *Graph Racers* and *Vector Rally*) is a two-player paper-and-pencil racing game that Jeff played on the bus in 5th grade.<sup>1</sup> The game is played with a track drawn on a sheet of graph paper. The players alternately choose a sequence of grid points that represent the motion of a car around the track, subject to certain constraints explained below.

Each car has a *position* and a *velocity*, both with integer  $x$ - and  $y$ -coordinates. The initial position is a point on the starting line, chosen by the player; the initial velocity is always  $(0, 0)$ . At each step, the player optionally increments or decrements either or both coordinates of the car's velocity; in other words, each component of the velocity can change by **at most 1** in a single step. The car's new position is then determined by adding the new velocity to the car's previous position. The new position must be inside the track; otherwise, the car crashes and that player loses the race. The race ends when the first car reaches a position on the finish line.

Suppose the racetrack is represented by an  $n \times n$  array of bits, where each 0 bit represents a grid point inside the track, each 1 bit represents a grid point outside the track, the 'starting line' is the first column, and the 'finish line' is the last column.

Describe and analyze an algorithm to find the minimum number of steps required to move a car from the starting line to the finish line of a given racetrack. [Hint: Build a graph. What are the vertices? What are the edges? What problem is this?]

velocity	position
(0, 0)	(1, 5)
(1, 0)	(2, 5)
(2, -1)	(4, 4)
(3, 0)	(7, 4)
(2, 1)	(9, 5)
(1, 2)	(10, 7)
(0, 3)	(10, 10)
(-1, 4)	(9, 14)
(0, 3)	(9, 17)
(1, 2)	(10, 19)
(2, 2)	(12, 21)
(2, 1)	(14, 22)
(2, 0)	(16, 22)
(1, -1)	(17, 21)
(2, -1)	(19, 20)
(3, 0)	(22, 20)
(3, 1)	(25, 21)



A 16-step Racetrack run, on a  $25 \times 25$  track.

4. A palindrome is any string that is exactly the same as its reversal, like I, or DEED, or RACECAR, or AMANAPLANACATACANALPANAMA. Describe and analyze an algorithm to find the length of the longest *subsequence* of a given string that is also a palindrome.

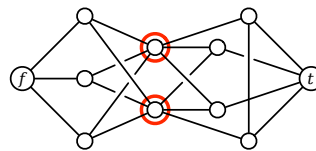
For example, the longest palindrome subsequence of MAHDYNAMICPROGRAMZLETMESHOWYOUTHEM is MHYMRORMYHM, so given that string as input, your algorithm should output the number 11.

<sup>1</sup>The actual game is a bit more complicated than the version described here.

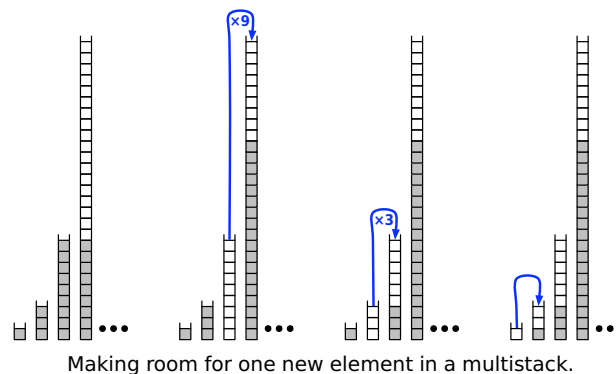
5. The Island of Sodor is home to a large number of towns and villages, connected by an extensive rail network. Recently, several cases of a deadly contagious disease (either swine flu or zombies; reports are unclear) have been reported in the village of Ffarquhar. The controller of the Sodor railway plans to close down certain railway stations to prevent the disease from spreading to Tidmouth, his home town. No trains can pass through a closed station. To minimize expense (and public notice), he wants to close down as few stations as possible. However, he cannot close the Ffarquhar station, because that would expose him to the disease, and he cannot close the Tidmouth station, because then he couldn't visit his favorite pub.

Describe and analyze an algorithm to find the minimum number of stations that must be closed to block all rail travel from Ffarquhar to Tidmouth. The Sodor rail network is represented by an undirected graph, with a vertex for each station and an edge for each rail connection between two stations. Two special vertices  $f$  and  $t$  represent the stations in Ffarquhar and Tidmouth.

For example, given the following input graph, your algorithm should return the number 2.



6. A *multistack* consists of an infinite series of stacks  $S_0, S_1, S_2, \dots$ , where the  $i$ th stack  $S_i$  can hold up to  $3^i$  elements. Whenever a user attempts to push an element onto any full stack  $S_i$ , we first pop all the elements off  $S_i$  and push them onto stack  $S_{i+1}$  to make room. (Thus, if  $S_{i+1}$  is already full, we first recursively move all its members to  $S_{i+2}$ .) Moving a single element from one stack to the next takes  $O(1)$  time.



- (a) In the worst case, how long does it take to push one more element onto a multistack containing  $n$  elements?
- (b) **Prove** that the amortized cost of a push operation is  $O(\log n)$ , where  $n$  is the maximum number of elements in the multistack.
7. Recall the problem 3COLOR: Given a graph, can we color each vertex with one of 3 colors, so that every edge touches two different colors? We proved in class that 3COLOR is NP-hard.

Now consider the related problem 12COLOR: Given a graph, can we color each vertex with one of twelve colors, so that every edge touches two different colors? **Prove** that 12COLOR is NP-hard.

**You may assume the following problems are NP-hard:**

**CIRCUITSAT:** Given a boolean circuit, are there any input values that make the circuit output True?

**PLANARCIRCUITSAT:** Given a boolean circuit drawn in the plane so that no two wires cross, are there any input values that make the circuit output True?

**3SAT:** Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

**MAXINDEPENDENTSET:** Given an undirected graph  $G$ , what is the size of the largest subset of vertices in  $G$  that have no edges among them?

**MAXCLIQUE:** Given an undirected graph  $G$ , what is the size of the largest complete subgraph of  $G$ ?

**MINVERTEXCOVER:** Given an undirected graph  $G$ , what is the size of the smallest subset of vertices that touch every edge in  $G$ ?

**MINSETCOVER:** Given a collection of subsets  $S_1, S_2, \dots, S_m$  of a set  $S$ , what is the size of the smallest subcollection whose union is  $S$ ?

**MINHITTINGSET:** Given a collection of subsets  $S_1, S_2, \dots, S_m$  of a set  $S$ , what is the size of the smallest subset of  $S$  that intersects every subset  $S_i$ ?

**3COLOR:** Given an undirected graph  $G$ , can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

**HAMILTONIANCYCLE:** Given a graph  $G$ , can is there a cycle in  $G$  that visits every vertex once?

**HAMILTONIANPATH:** Given a graph  $G$ , can is there a path in  $G$  that visits every vertex once?

**DOUBLEHAMILTONIANCYCLE:** Given a graph  $G$ , can is there a closed walk in  $G$  that visits every vertex twice?

**DOUBLEHAMILTONIANPATH:** Given a graph  $G$ , can is there an open walk in  $G$  that visits every vertex twice?

**MINDEGREE SPANNING TREE:** Given an undirected graph  $G$ , what is the minimum degree of any spanning tree of  $G$ ?

**MINLEAVES SPANNING TREE:** Given an undirected graph  $G$ , what is the minimum number of leaves in any spanning tree of  $G$ ?

**TRAVELINGSALESMAN:** Given a graph  $G$  with weighted edges, what is the minimum cost of any Hamiltonian path/cycle in  $G$ ?

**LONGESTPATH:** Given a graph  $G$  with weighted edges and two vertices  $s$  and  $t$ , what is the length of the longest *simple* path from  $s$  to  $t$  in  $G$ ?

**SUBSETSUM:** Given a set  $X$  of positive integers and an integer  $k$ , does  $X$  have a subset whose elements sum to  $k$ ?

**PARTITION:** Given a set  $X$  of positive integers, can  $X$  be partitioned into two subsets with the same sum?

**3PARTITION:** Given a set  $X$  of  $n$  positive integers, can  $X$  be partitioned into  $n/3$  three-element subsets, all with the same sum?

**MINESWEEPER:** Given a Minesweeper configuration and a particular square  $x$ , is it safe to click on  $x$ ?

**TETRIS:** Given a sequence of  $N$  Tetris pieces and a partially filled  $n \times k$  board, is it possible to play every piece in the sequence without overflowing the board?

**SUDOKU:** Given an  $n \times n$  Sudoku puzzle, does it have a solution?

**KENKEN:** Given an  $n \times n$  Ken-Ken puzzle, does it have a solution?