CS 473: Undergraduate Algorithms, Spring 2009
Head Banging Session 0

January 20 and 21, 2009

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1. Solve the following recurrences. If base cases are provided, find an *exact* closed-form solution. Otherwise, find a solution of the form $\Theta(f(n))$ for some function f.

   - **Warmup:** You should be able to solve these almost as fast as you can write down the answers.
     (a) $A(n) = A(n-1) + 1$, where $A(0) = 0$.
     (b) $B(n) = B(n-5) + 2$, where $B(0) = 17$.
     (c) $C(n) = C(n-1) + n^2$
     (d) $D(n) = 3\,D(n/2) + n^2$
     (e) $E(n) = 4\,E(n/2) + n^2$
     (f) $F(n) = 5\,F(n/2) + n^2$

   - **Real practice:**
     (a) $A(n) = A(n/3) + 3A(n/5) + A(n/15) + n$
     (b) $B(n) = \min_{0<k<n} (B(k) + B(n-k) + n)$
     (c) $C(n) = \max_{n/4<k<3n/4} (C(k) + C(n-k) + n)$
     (d) $D(n) = \max_{0<k<n} \big(D(k) + D(n-k) + k(n-k)\big)$, where $D(1) = 0$
     (e) $E(n) = 2E(n-1) + E(n-2)$, where $E(0) = 1$ and $E(1) = 2$
     (f) $F(n) = \dfrac{1}{F(n-1)\,F(n-2)}$, where $F(0) = 1$ and $F(2) = 2$
     *(g) $G(n) = n\,G(\sqrt{n}) + n^2$

2. The *Fibonacci numbers* F_n are defined recursively as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for every integer $n \geq 2$. The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$.

   Prove that any non-negative integer can be written as the sum of distinct *non-consecutive* Fibonacci numbers. That is, if any Fibonacci number F_n appears in the sum, then its neighbors F_{n-1} and F_{n+1} do not. For example:

$$\begin{aligned} 88 &= 55 + 21 + 8 + 3 + 1 &&= F_{10} + F_8 + F_6 + F_4 + F_2 \\ 42 &= 34 + 8 &&= F_9 + F_6 \\ 17 &= 13 + 3 + 1 &&= F_7 + F_4 + F_2 \end{aligned}$$

3. Whenever groups of pigeons gather, they instinctively establish a *pecking order*. For any pair of pigeons, one pigeon always pecks the other, driving it away from food or potential mates. The same pair of pigeons always chooses the same pecking order, even after years of separation, no matter what other pigeons are around. Surprisingly, the overall pecking order can contain cycles—for example, pigeon A pecks pigeon B, which pecks pigeon C, which pecks pigeon A.

   Prove that any finite set of pigeons can be arranged in a row from left to right so that every pigeon pecks the pigeon immediately to its left. Pretty please.