Proving that a problem $X$ is NP-hard requires several steps:

- Choose a problem $Y$ that you already know is NP-hard (because we told you so in class).
- Describe an algorithm to solve $Y$, using an algorithm for $X$ as a subroutine. Typically this algorithm has the following form: Given an instance of $Y$, transform it into an instance of $X$, and then call the magic black-box algorithm for $X$.
- ***Prove*** that your algorithm is correct. This always requires two separate steps, which are usually of the following form:
  - ***Prove*** that your algorithm transforms "good" instances of $Y$ into "good" instances of $X$.
  - ***Prove*** that your algorithm transforms "bad" instances of $Y$ into "bad" instances of $X$. Equivalently: Prove that if your transformation produces a "good" instance of $X$, then it was given a "good" instance of $Y$.
- Argue that your algorithm for $Y$ runs in polynomial time. (This is usually trivial.)

---

1. Recall the following $k$Color problem: Given an undirected graph $G$, can its vertices be colored with $k$ colors, so that every edge touches vertices with two different colors?

   (a) Describe a direct polynomial-time reduction from 3Color to 4Color.

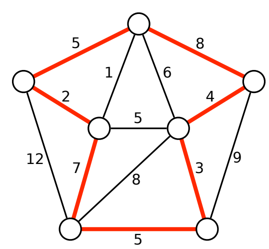
   (b) Prove that $k$Color problem is NP-hard for any $k \geq 3$.

2. A *Hamiltonian cycle* in a graph $G$ is a cycle that goes through every vertex of $G$ exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

   A ***tonian cycle*** in a graph $G$ is a cycle that goes through at least *half* of the vertices of $G$. Prove that deciding whether a graph contains a tonian cycle is NP-hard.

**To think about later:**

3. Let $G$ be an undirected graph with weighted edges. A Hamiltonian cycle in $G$ is ***heavy*** if the total weight of edges in the cycle is at least half of the total weight of all edges in $G$. Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.




A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.