

General Instructions: Please typeset your homework using \LaTeX , with each problem starting on a new page. See the course web site for advice on producing and importing figures. *Stars indicate problems that I don't already know how to solve; these problems may or may not be open.

I *strongly* encourage you to work together, but each student must turn in their own solutions. Similarly, you may use any resource at your disposal—human, printed, or electronic—but you must not copy anything verbatim, and you must cite *every* source you use, including the other students you work with. In other words, follow the same standards of scholarship you would use for a research paper.

1. Any set of points in the plane implicitly defines a weighted, complete graph whose vertices are the points themselves, whose edges are line segments, and where the weight of an edge is its Euclidean length. A *Euclidean minimum spanning tree* of a set of points is a minimum spanning tree of this graph.
 - (a) Describe a set of points in the plane with more than one Euclidean minimum spanning tree.
 - (b) Prove that for any Euclidean minimum spanning tree of any point set P in the plane, the maximum degree of any point is 6.
 - (c) Prove that every point set in the plane has a Euclidean minimum spanning tree in which the maximum degree is 5. In particular, if the Euclidean minimum spanning tree is unique, it has maximum degree 5.
 - (d) Prove that for any point set P in the plane, every Euclidean minimum spanning tree of P is a subgraph of the Delaunay graph of P .
 - (e) Describe an algorithm to compute the Euclidean minimum spanning tree of a set of n points in the plane in $O(n \log n)$ time. [Hint: Your solution should be very short!]

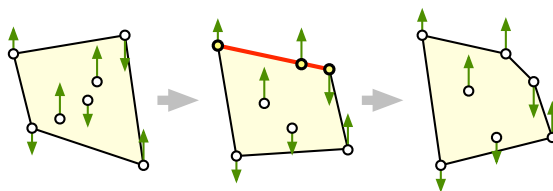
2. Consider a set P of n points in the plane, each moving along a vertical line at some fixed velocity. Each moving point p_i in P is specified by an ordered triple (x_i, y_i, \dot{y}_i) , indicating that at time t , point p_i has coordinates $(x_i, y_i + \dot{y}_i t)$. In other words,

$$p_i(t) = (x_i, y_i + \dot{y}_i t).$$

Let $P(t)$ denote the set of all points in P at time t . The convex hull of $P(t)$ varies continuously as a function of t . However, the *combinatorial structure* of the convex hull—which points lie on the hull, and in what order—changes only at certain discrete values of t , when some point $p_i(t)$ lies on a convex hull edge $p_j(t)p_k(t)$.

- (a) Prove that the convex hull of any n vertically-moving points undergoes only $O(n)$ combinatorial changes.
- (b) Prove that for any integer n , there is a set of n vertically-moving points whose convex hull undergoes $\Omega(n)$ combinatorial changes.

For full credit, the upper bound you prove in part (a) and the lower bound you prove in part (b) should match *exactly*. [Hint: This problem is easier than it looks. What else could the triple (x_i, y_i, \dot{y}_i) represent?]



The convex hull of seven vertically-moving points. The middle image shows a combinatorial change.

3. Let Π be a convex polygon in the plane. Describe and analyze a randomized incremental algorithm to construct the Delaunay triangulation of the vertices of Π in $O(n)$ expected time, given the convex polygon Π as input. [Hint: Delete the points in random order, then reinsert them in the opposite order.]
4. For each of the following primitives, describe a set of n points in the plane that is non-degenerate with respect to that primitive, where n is an arbitrary positive integer. Your points should have integer coordinates, each at most $O(n^c)$ for some constant c . If necessary, describe your set as the output of an algorithm that takes the integer n as input. Prove your answers are correct.

- (a) COMPARE_SLOPE(p, q, r, s): Is the slope of the line through points p and q greater than, equal to, or less than the slope of the line through points r and s ? Equivalently, is the following expression positive, zero, or negative?

$$(p_2 - q_2)(r_1 - s_1) - (p_1 - q_1)(r_2 - s_2)$$

[Hint: Choose a set of points on the parabola $y = x^2$.]

- (b) COMPARE_AREA(p, q, r, s, t, u): Is the area of Δpqr greater than, equal to, or less than the area of Δstu ? Equivalently, is the following expression positive, zero, or negative?

$$\begin{vmatrix} 1 & p_1 & p_2 \\ 1 & q_1 & q_2 \\ 1 & r_1 & r_2 \end{vmatrix} - \begin{vmatrix} 1 & s_1 & s_2 \\ 1 & t_1 & t_2 \\ 1 & u_1 & u_2 \end{vmatrix}$$

[Hint: Choose a set of points on the parabola $y = x^2$.]

- (c) IN_CONIC?(p, q, r, s, t, u): Does point u lie inside, on, or outside the unique conic (circle, ellipse, parabola, hyperbola, etc.) passing through points $p, q, r, s,$ and t ? Equivalently, is the following determinant positive, zero, or negative?

$$\begin{vmatrix} 1 & p_1 & p_2 & p_1^2 & p_1 p_2 & p_2^2 \\ 1 & q_1 & q_2 & q_1^2 & q_1 q_2 & q_2^2 \\ 1 & r_1 & r_2 & r_1^2 & r_1 r_2 & r_2^2 \\ 1 & s_1 & s_2 & s_1^2 & s_1 s_2 & s_2^2 \\ 1 & t_1 & t_2 & t_1^2 & t_1 t_2 & t_2^2 \\ 1 & u_1 & u_2 & u_1^2 & u_1 u_2 & u_2^2 \end{vmatrix}$$

[Hint: The parabola won't work here. Use another curve and Descartes' rule of sign.]

- * (d) Describe a set of n points in the plane that is non-degenerate with respect to all three of these primitives.