2 Winding Numbers

2.1 Let me not be pent up, sir; I will fast, being loose.

Fast and Loose is the name of a family of magic tricks (or con games) with ropes, chains, and belts that have been practiced since at least the 14th century; the con game is mentioned in three different Shakespeare plays. !In one such trick (now sometimes called the *Endless Chain*), the con artist arranges a closed loop of chain into a figure-8, and then asks the mark to put their finger on the table inside one of the loops. The con artist them pulls the chain along the table. If the chain catches on the mark's finger, then the chain is *fast* and the mark wins; if the con artist can pull the chain completely off the table, the chain is *loose* and the mark loses.

The con artist shows the mark that there are two different ways for the loops to fall. (Notice how the chain crosses itself in the lower corners.) Because the chain is bright and shiny and bumpy, it's impossible for the mark to tell which way the chain is actually arranged, but because these are the only possibilities, the mark should have a 50-50 chance of winning. Right? Right?

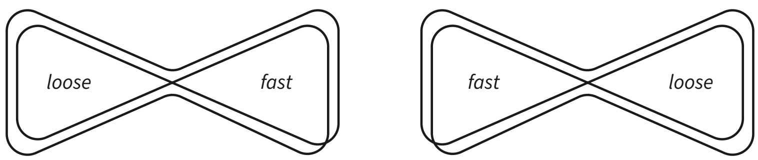

Figure 1: Two arrangements of the Endless Chain

Of course not! As soon as the mark places money on the barrelhead, the con artist wins every time. The con artists was lying; there is a third arrangement of the chain that is *always* loose, no matter where the mark puts their finger.

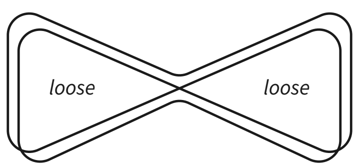

Figure 2: The actual arrangement of the Endless Chain

2.2 Shoelaces and Signed Areas

[Write this!]

```
def signedArea(P):
    area = 0
    n = size(P)
    for i in range(n):
        area += (P[i].x * P[(i+1)%n].y - P[i].y * P[(i+1)%n].x) / 2
    return area
```

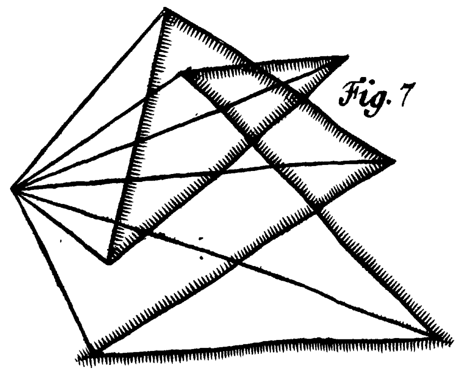

Figure 3: Computing the signed area of a polygon, from Meister (1785)

2.3 Winding numbers

The *winding number* of a polygon P around a point o is intuitively (and not surprisingly) the number of times that P winds counterclockwise around o. For example, if P is a simple polygon, its winding number around any exterior point is zero, and its winding number around any interior point is either $+1$ or -1, depending on how the polygon is oriented. If the polygon winds *clockwise* around o, the winding number is negative. Crucially, the winding number is only well defined if the polygon does not *contain* the point o.

We can define the winding number more precisely in terms of angles as follows. Let $p_0, p_1, \ldots, p_{n-1}$ denote the vertices of P in order. For each index i, let θ_i denote the interior angle at o in the triangle $\triangle p_i o p_{i+1}$, with positive sign if $(0, p_i, p_{i+1})$ is oriented counterclockwise, and with negative sign if $(0, p_i, p_{i+1})$ is oriented clockwise. Assuming angles are measured in *circles* (the way the gods intended, as opposed to radians or degrees or some other idiocy), the winding number of P around o is the sum $\sum_i \theta_i$.

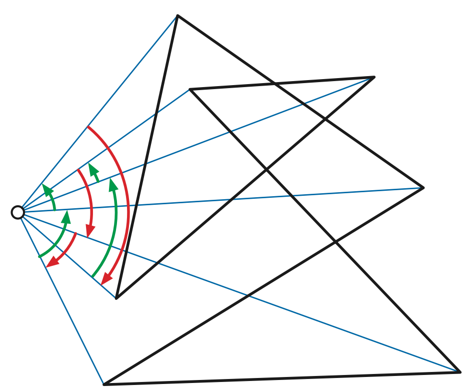

Figure 4: Winding number as a sum of angles, after Meister

Actually computing the winding number according to this definition requires inverse trigonometric functions, square roots, and other numerical madness. Fortunately, there is an equivalent definition that builds on our ray-shooting test from the previous lecture. Let R be a vertical ray shooting upward from o. We distinguish two types of crossings between the R and the polygon, depending on the orientation of the crossed edges. Specifically, if the crossed edge is directed from right to left, we have a *positive* crossing; otherwise, we have a *negative* crossing.

Equivalently, when R crosses an edge $p_i p_{i+1}$, the sign of the crossing is the sign of the determinant $\Delta(o, p_i, p_{i+1})$.

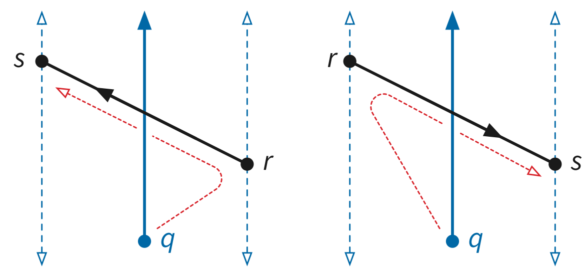

Figure 5: A positive crossing (left) and a negative crossing (right)

I'll leave the equivalence of these two definitions as an exercise. (Hint: prove equivalence for triangles, and then look at Meister's figure again!)

Here is the ray-shooting algorithm in (psuedo)Python. Any similarities with the point-in-polygon algorithm from the previous lecture are purely intentional.

```
def windingNumber(P, o):
    wind = 0
    n = size(P)
    for i in range(n):
        p = P[i]
        q = P[(i+1)%n]
        Delta = (p.x - o.x)*(q.y - o.y) - (p.y - o.y)*(q.x - o.x)
        if p.x <= o.x < q.x && Delta > 0:
            wind += 1
        elif q.x <= o.x < p.x && Delta < 0:
            wind -= 1
    return wind
```

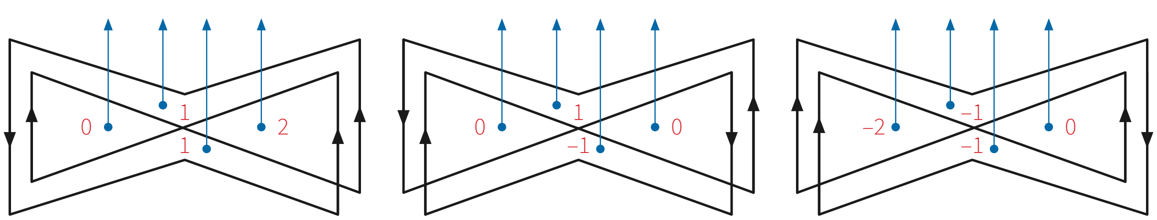

Figure 6: Winding numbers of the Endless Chain around various points

2.4 Homotopy

A *homotopy* between two closed curves is a continuous deformation—a morph—from one curve to the other. Homotopies can be defined between curves in any topological space, but for purposes of illustration, let's restrict ourselves to curves in the punctured plane $\mathbb{R}^2 \setminus o$, where o is an arbitrary point called the *obstacle*.

Formally, a *free homotopy* between two closed curves in $\mathbb{R}^2 \setminus o$ is a continuous function $h\colon [0,1] \times S^1 \to \mathbb{R}^2 \setminus o$, such that $h(0,\cdot)$ and $h(1,\cdot)$ are the initial and final closed curves, respectively. For each $0 < t < 1$, the function $h(t, \cdot)$ is the intermediate closed curve at "time" t. Crucially, none of these intermediate curves touches the obstacle point o.

Two closed curves in $\mathbb{R}^2 \setminus o$ are *homotopic*, or in the same *homotopy class*, if there is a homotopy from one to the other in $\mathbb{R}^2 \setminus o$. Homotopy is an equivalence relation.

A closed curve is *contractible* in $\mathbb{R}^2 \setminus o$ if it is homotopic to a single point (or more formally, to a constant curve).

We also need a definition of homotopy between *paths*; this is a little more subtle. Let $\pi\colon [0,1] \to \mathbb{R}^2 \setminus o$ and $\sigma\colon [0,1] \to \mathbb{R}^2 \setminus o$ be two paths in the punctured plane with the same endpoints: $\pi(0) = \sigma(0)$ and $\pi(1) = \sigma(1)$. A *path homotopy* from π to σ is a continuous function $h\colon [0,1] \times [0,1] \to \mathbb{R}^2 \setminus o$ that satisfies four conditions:

- $H(0,t) = \pi(t)$ for all t
- $H(1,t) = \sigma(t)$ for all t
- $H(s,0) = \pi(0) = \sigma(0)$ for all s
- $H(s,1) = \pi(1) = \sigma(1)$ for all s

Intuitively, you should think of a path homotopy as a continuous deformation of one path into the other, keeping the endpoints fixed at all times. Again, for each $0 < s < 1$, the function $h(t, \cdot)$ is the intermediate path at "time" s, and none of these intermediate paths touches the obstacle point o.

I'll typically use the word "homotopy" for both free homotopy and path homotopy, in the hope that the precise type is clear from context.

2.5 Vertex moves

Similar to the definition of "connected", the definition of "homotopy" allows intermediate curves to be arbitrarily wild closed curves even if the initial and final curves are polygons.

Fortunately, there is a general principle that allows us to "tame" homotopies between tame curves like polygons, by decomposing them into a sequence of elementary *moves*. (This principle is similar to the observation that any closed curve can be approximated by a sequence of line segments, otherwise known as a polygon.)

Let P be any polygon. A *vertex move* translates exactly one point p of P along a straight line from its current location to a new location p', yielding a new polygon P'. You should imagine that as the point p moves, the edges incident to p pivot around their other endpoints. Typically the moving point p is a vertex of the initial polygon P and the final point p' is a vertex of the final polygon P', but neither of these restrictions is required by the definition. We are allowed to freely introduce new vertices in the middle of edges, or freely delete "flat" vertices between two collinear edges.

Figure!

Now suppose the polygon P lives in the punctured place $\mathbb{R}^2 \setminus o$. Let p, q, r be three consecutive vertices of P. The vertex move $q \mapsto q'$ is *safe* if neither of the triangles $\triangle pqq'$ or $\triangle qq'r$. contains the obstacle point o. Equivalently, during a safe vertex move, the continuously changing polygon never touches o.

It follows that every safe vertex move is a homotopy in $\mathbb{R}^2 \setminus o$. We can build up more complex homotopies by concatenating several safe vertex moves. In fact, *any* sequence of safe vertex moves describes a homotopy in $\mathbb{R}^2 \setminus o$

2.6 Polygon homotopies are sequences of vertex moves

Unfortunately, the converse of this observation is false; not every homotopy is a sequence of vertex moves. Consider, for example, a simple translation or rotation of the entire polygon! Nevertheless, every homotopy can be *approximated* by a sequence of safe vertex moves.

Lemma: If two polygons in $\mathbb{R}^2 \setminus o$ are homotopic, then they are homotopic by a sequence of safe vertex moves.

Proof: Fix a homotopy $h\colon [0,1]\times S^1 \to \mathbb{R}^2 \setminus o$ between two polygons $P_0 = h(0,\cdot)$ and $P_1 = h(1,\cdot)$.

For any parameters t and θ, let $d(t,\theta)$ be the Euclidean distance from $h(t,\theta)$ to the origin o, and let $\varepsilon = \min_{t,\theta} d(t,\theta)$. Because the cylinder $[0,1]\times S^1$ is compact, this minimum is well-defined and positive.

We subdivide the cylinder $[0,1]\times S^1$ into triangles as follows. First, cut the cylinder into a grid of $\delta \times \delta$ squares $\square(i,j) = [i\delta,(i+1)\delta] \times [j\delta,(j+1)\delta \bmod 1]$, where $\delta > 0$ is chosen so that the diameter of $h(\square(i,j))$ is at most $\varepsilon/2$. (The existence of δ is guaranteed by continuity. Then further subdivide each grid square into two right isosceles triangles; see Figure 7. Without loss of generality, assume each vertex of P_0 and P_1 the image of some vertex on the boundary of the resulting triangle mesh Δ.

The homotopy h maps any cycle in this grid to a closed curve, which consists of $O(1/\delta)$ curve segments, each with diameter at most $\varepsilon/2$, and each with distance at least ε from o. Define a new homotopy $h'\colon [0,1]\times S^1 \to \mathbb{R}^2 \setminus o$ that agrees with h at every grid vertex and linearly interpolates within each grid triangle. Changing from h to h' changes the image of any grid cycle by replacing each short curve segment with a straight line segment.

We can easily construct a sequence of $1 + 2/\delta^2$ cycles in Δ that starts with one boundary $0 \times S^1$ and ends with the other boundary $1 \times S^1$, such that the symmetric difference between two adjacent cycles is the boundary of one triangle in Δ. Two adjacent cycles in this sequence is shown on the right in Figure 7. The piecewise-linear homotopy h' maps any two adjacent cycles in this sequence to a pair of polygons that differ by a vertex move.

Thus, we obtain a sequence of $1+2/\delta^2$ vertex moves transforming P_0 into P_1. Every vertex of each intermediate polygon has distance at least ε from the origin, each edge has length at most $\varepsilon/2$, and each vertex move translates its vertex a distance of at most $\varepsilon/2$. It follows that every vertex move in this sequence is safe.

This lemma is a special case of a more general *simplicial approximation theorem*, which intuitively states that any continuous map between nice topological spaces can be approximated by a nice continuous map; moreover, the original map can be continuously deformed to its approximation.

2.7 Homotopy Invariant

Winding numbers are our first example of a *topological invariant*, and specifically a *complete homotopy* invariant. A topological invariant is any property of objects or spaces that is unchanged by homeomorphism; a standard example for connected orientable surfaces is the *genus*. A

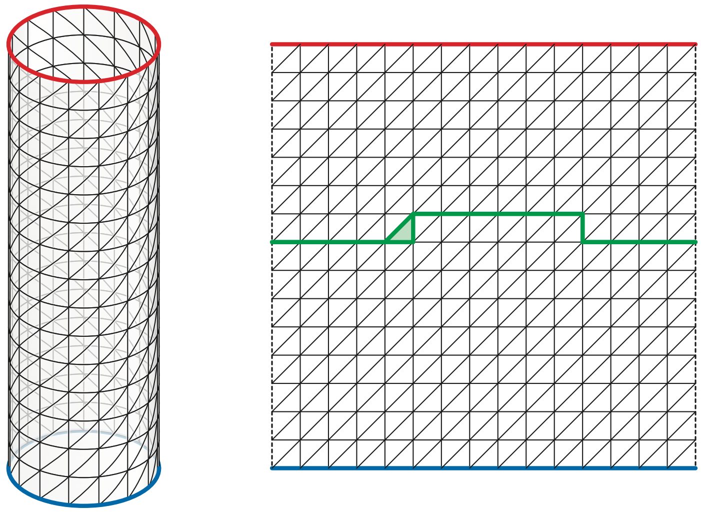

Figure 7: A grid on the unit cylinder.

homotopy invariant is any property that is preserved by homotopy; a homotopy invariant is *complete* if it takes on different values for two objects that are not homotopic.

Theorem: Two polygons are homotopic in $\mathbb{R}^2 \setminus o$ if and only if they have the same winding number around the origin o.

Proof: Fix two polygons P_0 and P_1 in $\mathbb{R}^2 \setminus o$. If these two polygons are homotopic, then by the previous lemma, they are connected by a sequence of safe triangle moves. A safe triangle move does not change the winding number of a polygon around the origin. Thus, by induction, P_0 and P_1 have the same winding number.

To prove the converse, I'll describe a sequence of safe triangle moves that transforms any polygon P into a *canonical* polygon $\Diamond^w$ with the same winding number w around the origin. (The notation $\Diamond^w$ will make sense later, honest.) Thus, if P_0 and P_1 have the same winding number w, we can deform P_0 into P_1 by concatenating the move sequence that takes P_0 to $\Diamond^w$ and the reverse of the move sequence that takes P_1 to $\Diamond^w$.

Our homotopy consists of several stages. First let's consider the case where the winding number of P around 0 is not zero.

- Let p_i be any vertex of P, and let p_{i-1} and p_{i+1} be the next and previous vertices. We call q *redundant* if the triangle $\triangle p_{i-1}p_ip_{i+1}$ does not contain the origin. In particular, if the triples (o, p_{i-1}, p_i) and (o, p_i, p_{i+1}) have opposite orientations, one clockwise and the other counterclockwise, then p_i is redundant. In the first phase of our homotopy, we repeatedly remove redundant vertices, by moving each redundant vertex q to one of its neighbors, until none are left. The resulting polygon P' is *angularly monotone*: every triple (o, p_i, p_{i+1}) has the same orientation.
- Next, we subdivide P' by adding vertices at its intersections with rays pointing up, down, left, and right from the origin o. After this subdivision, any vertex that is *not* on one of these rays is redundant. So in the second phase of the homotopy, we remove all non-ray vertices using safe vertex moves. The resulting polygon P'' is still angularly monotone.

- Finally, we move each vertex so that is distance from the origin is 1; each of these vertex moves is safe. The resulting polygon $\diamond^w$ has vertices only at the points $(0,1)$, $(1,0)$, $(0,-1)$, and $(-1,0)$; the polygon winds around this diamond $|w|$ times, counterclockwise if $w > 0$ and clockwise is $w < 0$.

The special case where P has winding number 0 is even simpler. The first phase (removing redundant vertices) actually reduces P to a single point; we can then translate this point to $\diamond^0 = (1,0)$ using one more safe vertex move.

Figures!!

This theorem immediately implies a linear-time algorithm to decide if two polygons are homotopic in the punctured plane: Count how many times each polygon crosses an arbitrary ray from the origin in each direction.

2.8 ...and the Aptly Named Sir Not Appearing in This Film

- rotation number = total turning angle = smiles − frowns
- regular homotopy = vertex moves without spurs
- rotation number is a regular homotopy invariant
- complex root finding
- signed volumes of self-intersecting polyhedra (hic utres unilaterales nascuntur)