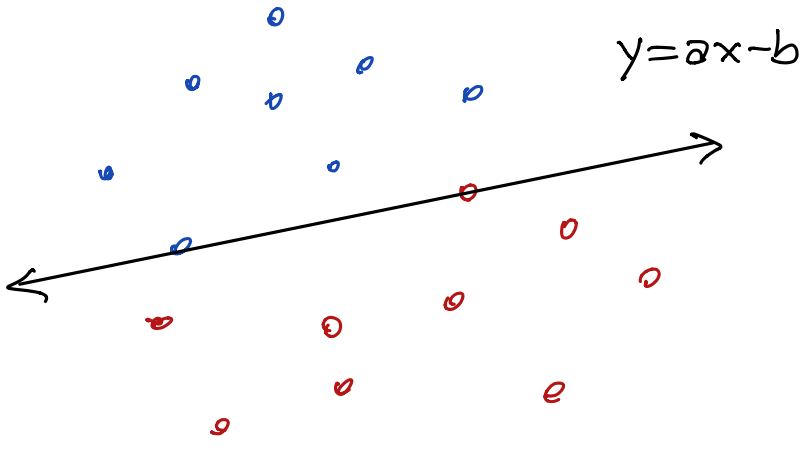
Linear Programming

Joe Jacksen – "Cancer"
Eels – "Hospital Food"

Example 1: Classification

Find a line with all blue points above red points below

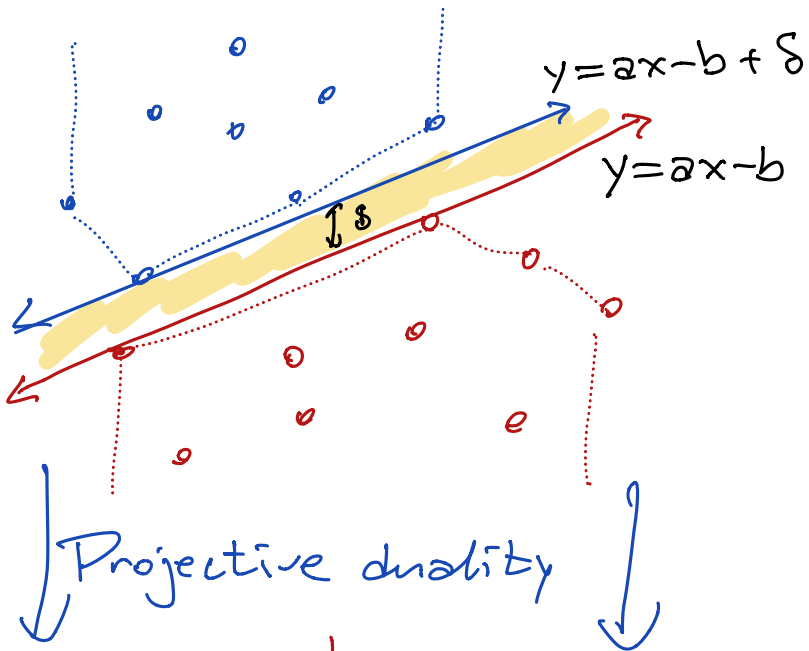


LP feasibility:

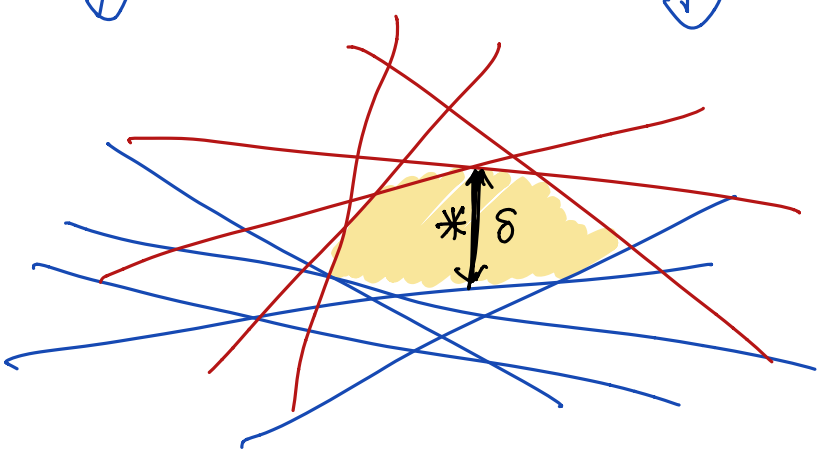
Find $(a, b) \in \mathbb{R}^2$
such that
$y_i \geq a x_i - b$ for every blue point (x_i, y_i)
$y_i \leq a x_i - b$ for every red pt (x_i, y_i)

Linear optimization (LP)

Find $(a, b, \delta) \in \mathbb{R}^3$
such that
$y_i \geq a x_i - b + \delta$
$y_i \leq a x_i - b$
$\delta \geq 0$
and δ is as large as possible

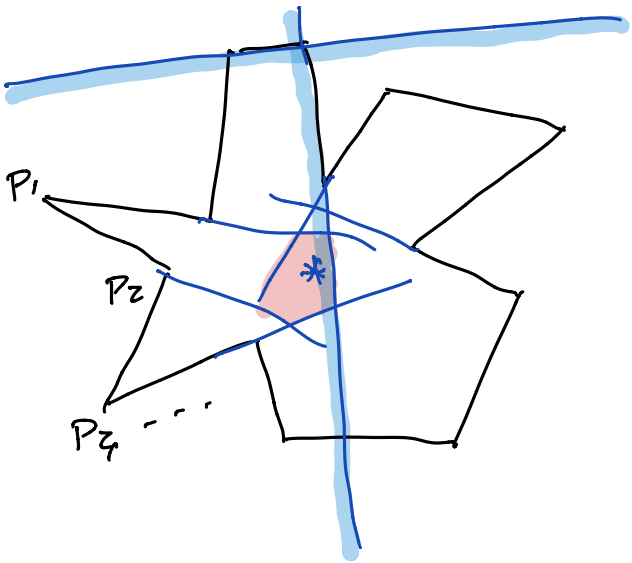


Projective duality



In fact, we can solve this problem in $O(n)$ time

We can build the polygon in $O(n \log n)$ time
And then find optimal δ in ~~$O(n)$ time~~ $O(\log n)$ time



A polygon is star-shaped if there is an interior point from which all edges are fully visible.

The set of all such points is the kernel of the polygon

= intersection of halfplanes.

= convex polygon or $\emptyset$

LP Feasibility:

Find $(x, y) \in \mathbb{R}^2$

such that $\det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_{i+1} & y_{i+1} \\ 1 & x & y \end{bmatrix} \geq 0$ for all i

$$\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix} - x \begin{vmatrix} 1 & y_i \\ 1 & y_{i+1} \end{vmatrix} + y \begin{vmatrix} 1 & x_i \\ 1 & x_{i+1} \end{vmatrix} \geq 0$$

General formulation: Symbolic

Given $A \in \mathbb{R}^{n \times d}$ (constraint matrix) $b \in \mathbb{R}^n$ (offset vector) $c \in \mathbb{R}^d$ (objective vector) n = # constraints

Find a vector $x \in \mathbb{R}^d$ d = dimension

such that $A_i \cdot x \leq b_i$ for all i (ith row)

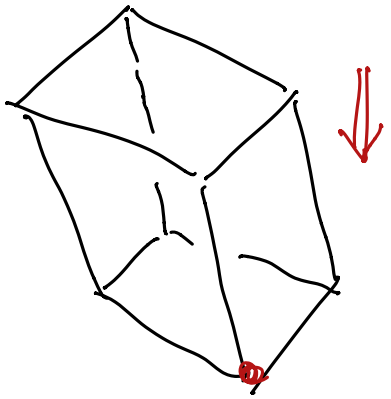
and $c \cdot x$ is as large as possible

Abbreviated:

$$\max \; c \cdot x \quad \text{s.t.} \quad Ax \leq b$$

Geometric:

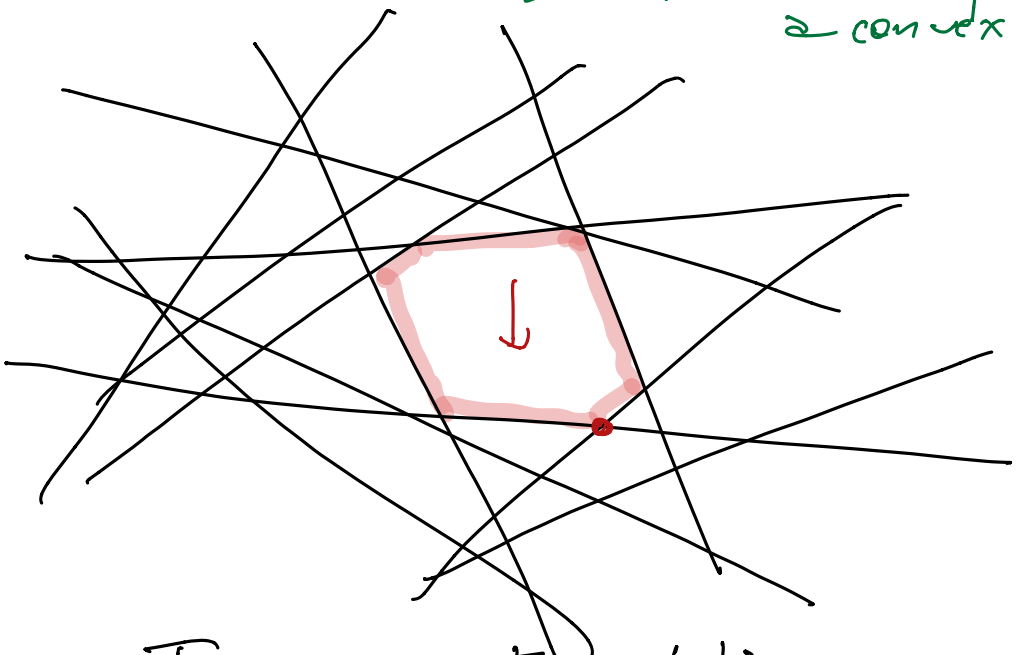
Every linear inequality $A_i \cdot x \leq b_i$ describe a halfspace in $\mathbb{R}^d$



Feasible points in $\mathbb{R}^d$
= intersection of halfspaces
= convex polyhedron

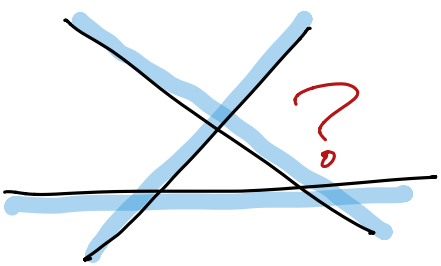
Objective vector = direction
WLOG down
$(0, 0, 0, \ldots, 0, -1)$

LP: Find lowest point inside a convex polyhedron

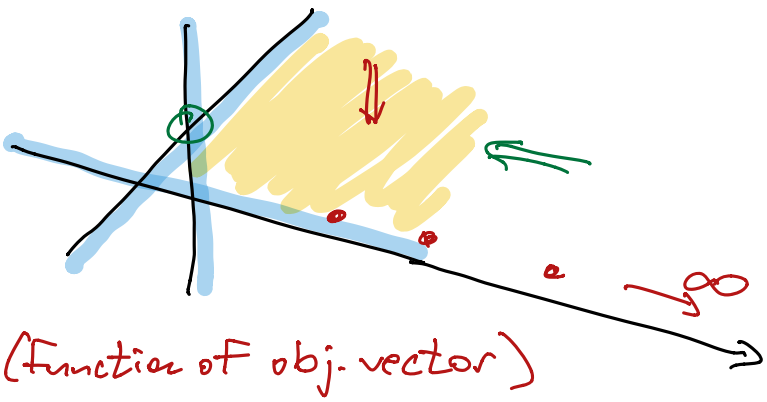


Two reasons the solution may not exist

Infeasible:

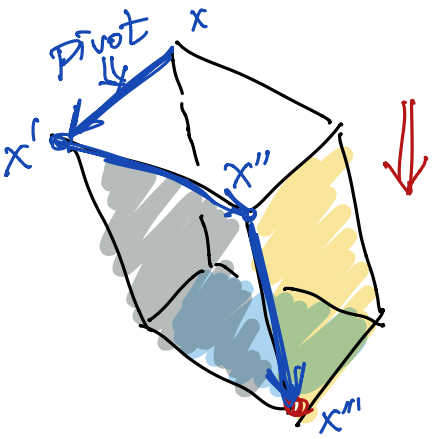


Unbounded



(function of obj. vector)

LP structure



Assuming general position ---

If LP is feasible + bounded

some optimal solution is a vertex of the feasible polyhedron

= feasible point on exactly d constraint planes

= satisfies exactly d constraints with equality

"basis" ←

Simplex algorithm

Find any feasible vertex x

Greedily improve x by walking along edges of feasible polyhedron

Worst case*: Visit every feasible vertex $\Theta\left(\binom{n}{\lfloor d/2 \rfloor}\right)$

Even with careful pivoting: exponential # pivots
deterministic — as far as we know

RANDOMIZE → subexponential $2^{\sqrt{d}}$

OPEN: polynomial?

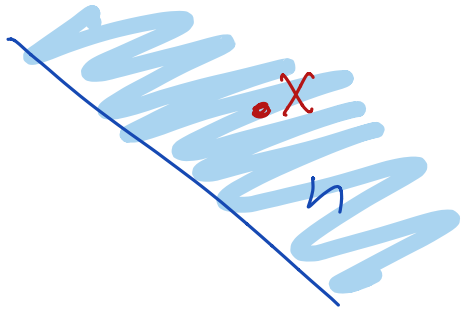
→ Fixed-parameter tractable:

If $d = O(1)$ then $O(n)$ time

SeidelLP(H, B):

```
if |H ∪ B| = d
    solve for x
    if x is feasible
        return x
    else
        return NO!
else
    choose h ∈ H at random
    x ← SeidelLP(H \ h, B)
    if x ∈ h or x = NO!
        return x
    else
        return SeidelLP(H \ h, B ∪ h)
```

H = halfspaces
= inequality constraints

B = partial basis = planes
= equality constraints

$H \cap B = \emptyset$