CS 498 TC ✧ Computational Geometry ✧ Spring 2026

Homework 2

Due Thursday, February 26, 2026 at 8pm

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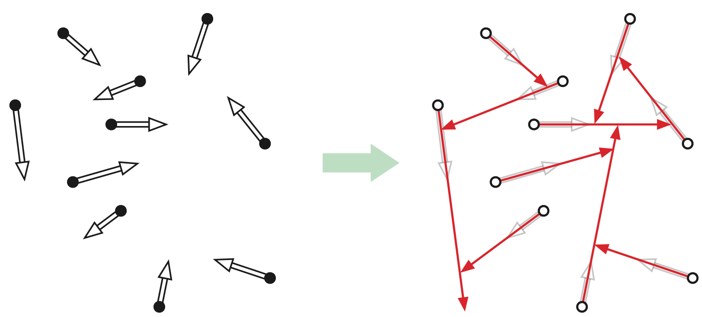
1. Suppose we are given a set M of n moving points in the plane, each with an initial position (x_i, y_i) and a fixed velocity (u_i, v_i). Under normal circumstances, at any time $t > 0$, the ith point in M has coordinates $p_i(t) = (x_i + t \cdot u_i, y_i + t \cdot v_i)$.

   However, if a moving point p_i ever reaches a location that another point has previously occupied, p_i stops moving forever. Similarly, if two moving points collide, by occupying the same location at the same time, they both stop forever. (But assuming general position, this never actually happens.)

   Eventually, each moving point either stops (by running into another points trail) or escapes off to infinity. The complete pattern of trails left by the points is called the *motorcycle graph* of M.[1]

   Ten moving points and their motorcycle graph.

   (a) Describe an algorithm to compute the motorcycle graph of n moving points.

   (b) Describe an algorithm to compute the motorcycle graph of n moving points e when $x_i = 0$ for every index i; that is, all n points start on the y-axis. This algorithm should be significantly faster than your algorithm for part (a).

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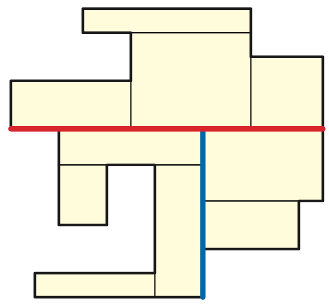
[1]Motorcycle graphs were inspired by the "light cycles" in the 1982 Disney movie *Tron*, except that the moving points can't turn, brake, or accelerate, and their jetwalls remain up forever, even after they derez.

2. A simple polygon is *orthogonal* if its edges alternate between horizontal and vertical. An orthogonal polygon is *generic* if no pair of edges lies on a common horizontal or vertical line. Let P be an arbitrary generic simple orthogonal polygon with n vertices.

   A *rectangulation* R of P is a partition of the interior of P into axis-aligned rectangles. (These rectangles do not necessarily meet edge-to-edge.) A *bar* in a rectangulation R is a maximal connected union of collinear edges in R; every edge of R is contained in exactly one bar. A rectangulation R of P is *proper* if (1) no vertex has degree 4 and (2) every bar in R contains an edge of the polygon P.



A proper rectangulation of a generic orthogonal polygon; the red and blue segments are bars.

   (a) Prove that P has exactly $n/2-2$ reflex vertices.

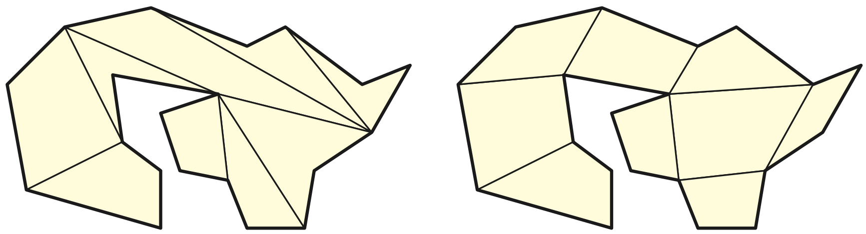
   (b) Prove that every proper rectangulation of P has exactly $n/2-1$ rectangles.

   (c) Describe an algorithm to construct a proper rectangulation of P. (In particular, this algorithm proves that a proper rectangulation always exists!)

   (d) Prove or disprove: In every proper rectangulation R of P, every rectangle in R touches the boundary of P.

   Think about later: Without the general-position assumption, and orthogonal polygon can have proper rectangulations with strictly less than $n/2-1$ rectangles. Describe an algorithm to compute a proper rectangulation with the minimum number of rectangles. *[Hint: You need some results from CS 473.]*

3. Recall that a (frugal) *triangulation* of a simple polygon P partitions the interior of P into triangles that meet edge-to-edge. Similarly, a *quadrangulation* of a polygon P partitions its interior into quadrilaterals that meet edge-to-edge. A *convex* quadrangulation is a quadrangulation in which every quadrilateral is convex. Just like triangulations, a quadrangulation of a polygon is *frugal* if every vertex of a quadrilateral is a vertex of the polygon.



Two frugal quadrangulations of the same simple polygon. Only the second quadrangulation is convex.

(a) Some easy warmup questions:

i. Prove that *every* simple polygon has a (not necessarily frugal) convex quadrangulation. *[Hint: There is a very short solution.]*

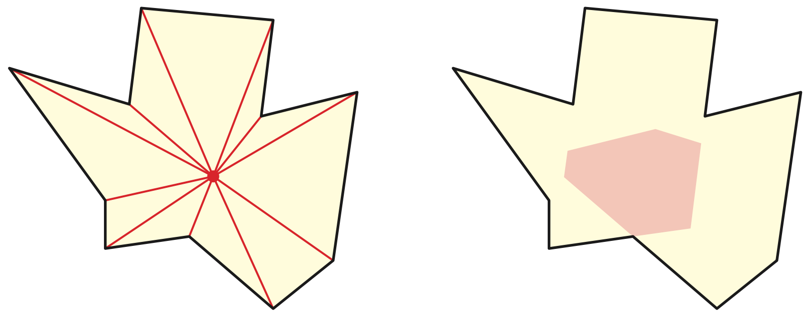
ii. Prove that every polygon that has a *frugal* quadrangulation has an even number of edges.

iii. Give an example of a simple polygon with an even number of vertices that has no frugal quadrangulation.

iv. Give an example of a simple polygon with an even number of vertices that has a frugal quadrangulation but has no frugal *convex* quadrangulation.

(b) Describe an algorithm that either computes a frugal quadrangulation of a given simple polygon, or correctly reports that no such quadrangulation exists. *[Hint: Dynamic programming]*

4. (a) A simple polygon P is *star-shaped* if there is an interior point g, such that for every vertex p of P, the line segment gp lies inside P. Such a point g is called a *guard*; informally, g can "see" every point inside P. The set of all guards is called the *kernel* of P.



One guard in a simple polygon, and the kernel of the same polygon.

Describe an algorithm to find a guard in an arbitrary simple polygon with at most five vertices in $O(1)$ time. This algorithm implies that every simple polygon with at most five vertices is star-shaped. *[Hint: The kernel of every polygon is convex.]*

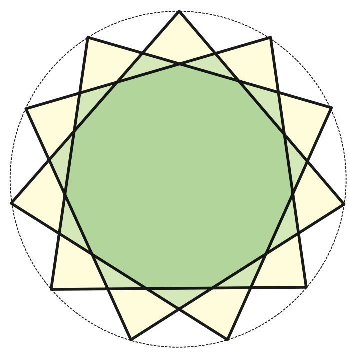
(b) Suppose we are given a doubly-connected edge list whose underlying graph G is simple and connected, that satisfies Euler's formula $V - E + F = 2$, and where every face in F has degree 3. (Formally, every face is a cycle of three distinct darts in G.)

Describe a linear-time algorithm to assign coordinates to the vertices of G to obtain a planar *straight-line* embedding consistent with the given PSLG. *[Hint: Invoke the Recursion Fáry, and then use part (a).]*

★5. **Open problem to think about later:** Suppose we are given a set P of n points in the plane. Describe an algorithm to compute a polygon vertices are precisely P and whose maximum weighted area (as computed by the shoelace formula) is as large as possible. Alternatively, prove that finding the maximum-weighted-area polygon is NP-hard.

I conjecture that even approximating the optimal polygon is NP-hard. This is a special case of the traveling salesman problem in a complete directed graph, where the weight of each edge $u \to v$ is the 2×2 determinant $u.x \cdot v.y - u.y \cdot v.x$; in particular, half of the edge weights are negative! However, as far as I know, this problem is completely open even in the extremely special case where the points P lie on a single circle.

For example, if P contains 11 evenly-spaced points on the unit circle, then the maximum-area polygon for P is a regular star polygon that winds three times around the origin. This polygon has weighted area $11 \cdot \sin(6\pi/11) \approx 10.88804$; for comparison, the convex hull of P has area $11 \cdot \sin(2\pi/11) \approx 5.94705$, and the unit circle has area $2\pi \approx 6.28319$.



The maximum-area polygon with eleven evenly-spaced cocircular vertices.