Elementary polygon things

Point in polygon

Given simple polygon $P$ and point $q$,
Is $q$ in the interior of $P$?

Intuition:

Split plane into slabs by vertical lines through vertices of $P$.

Edges of $P$ split slabs into trapezoids that alternate inside and outside.

Algorithm: Shoot a ray upward from $q$ and count crossings.
Even = out  Odd = in

Two types of crossings:

- Positive
  - $s \times q \times r \times x > 0$
  - $q$, $r$, $s$, $x$ are in clockwise order

- Negative
  - $s \times q \times r \times x < 0$
  - $q$, $r$, $s$, $x$ are in counterclockwise order

```
PtInPoly(P[0...n], q)
inside = False
for i = 0 to n-1
  r = P[i]
  s = P[(i+1) mod n]
  Delta = det \begin{bmatrix} 1 & q.x & q.y \\ 1 & r.x & r.y \\ 1 & s.x & s.y \end{bmatrix}
  if s.x < q.x < r.x and Delta > 0
    inside = ~inside
  else if r.x < q.x < s.x and Delta < 0
    inside = ~inside
return inside
```

Runs in $O(n)$ time

Winding Number
How many times does $P$ wind around $q$?

Sum of signed angles subtended by edges of $P$ around $q$.

But angles are EVIL!!!
Use the same ray-shooting algorithm:

- Count winds, don't sum angles.
- Works even for non-simple polygons!

**PtInPolygon** = [wind # is odd]

**Polygon area** (Gauss' shoelace algorithm)

Signed Area of a triangle

\[
\text{Area of a triangle } pqr = \frac{1}{2} \det \begin{bmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{bmatrix}
\]

Area of any simple polygon = sum of areas of \( \Delta s \) in any triangulation

But triangulation takes too long!
Cofactor expansion
\[
\begin{vmatrix}
1 & px & py \\
1 & qx & qy \\
1 & rx & ry \\
\end{vmatrix}
= \begin{vmatrix}
1 & px & py \\
1 & qx & qy \\
1 & rx & ry \\
\end{vmatrix}
- \begin{vmatrix}
1 & px & py \\
1 & qx & qy \\
1 & 0 & 0 \\
\end{vmatrix}
+ \begin{vmatrix}
1 & px & py \\
1 & qx & qy \\
1 & 0 & 0 \\
\end{vmatrix}
\]

\[
\text{Area}(pqr) = \text{Area}(oqr) + \text{Area}(orp) + \text{Area}(opq)
\]

Generalize to polygons via triangulation (diagonal Δs cancel out)

Polygon Area (P)

\[
\text{area} = \frac{1}{2} \det \begin{bmatrix} 1 & 0 & 0 \\ px & py & 0 \\ qx & qy & 0 \end{bmatrix}
\]

\frac{1}{2} (px\cdot qy - py\cdot qx)

Non simple polygons:
\[
\text{Area} = \sum_{\text{faces} f} \text{area}(f) \cdot \text{wind}(P, f)
\]

Start on planar graphs if there's time