Randomized incremental construction

$T_0 \leftarrow \text{empty decomposition}$

for $i = 1$ to $n$ in random order

insert $s_i$ into $T_{i-1} \rightarrow T_i$

last time cost to insert $s_i$

= time to locate $p_i$ + $O(\deg_i(s_i))$

$E[\text{cost insert } s_i] = E[\text{time to locate } p_i]$

+ $O(E(\deg_i(s_i)))$

Backward analysis

$S_i$ is a random segment in $T_i$

$E \deg(s) = O(i)$ by Euler's formula

so $E[\deg_i(s_i)] = O(1)$

$E[\text{time to locate } p_i] + O(1)$

$\Rightarrow$ Total expected time = $O(n)$

$+ E[\text{time for all } n \text{ point locations}]$

$\text{HOW?}$
Each trapezoid maintains the set of endpoints inside it. Each point maintains a trap that contains it.

When we split any trapezoid $\Delta$, we traverse $K(\Delta)$ by brute force and reassign every point.

Total time = $\sum_{\Delta \in T_{i-1} \setminus T_i} O(|K(\Delta)|)$ deleted traps

= $\sum_{\Delta \in T_i \setminus T_{i-1}} O(|K(\Delta)|)$ created traps

$X = \sum_j x_j$

$E[X] = E[\sum_j x_j]$

= $\sum_j E[x_j]$

= $\sum_j p[x_j = 1]$

$E[x_j] = O \cdot p[x_j = 0] + 1 \cdot p[x_j = 1]$

= $p[x_j = 1]$

Total over all iterations

Total exp. time = $O\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} \Pr[\Delta_i(p_j) \neq \Delta_{i-1}(p_j)]\right)$
\[
E \left[ \text{\# times trap containing } p_i \text{ changes} \right] \\
= O \left( \sum_{j=1}^{n} \sum_{i=1}^{j-1} \Pr \left[ \Delta_i(p_j) \neq \Delta_{i-1}(p_j) \right] \right) \\
= O \left( \sum_{i=1}^{n} \frac{H_i}{i} \right) \\
\leq O(n) \cdot \sum_{i=1}^{n} \frac{1}{i} = O(n \log n)
\]

Harmonic number \( H_n \)

\[ \ln(n+1) \leq H_n \leq [\ln n + 1] \]

**Special case:**

- Choose random segment
- Partition
- Recurse on both sides

Randomized quicksort!

Conflict list = recursive subproblem

History dag = binary search tree (treap)
History dag

Build a pt. loc. data structure

Init: empty

$\Omega \geq \left[ \text{# nodes per level} \right] = O(1) \Rightarrow E[\text{space}] = O(n)$

Depth per level $\leq 3$

Time to locate $q \leq 3 \times \# \text{levels where trap containing } q \text{ changes}$.

$E[\text{Time to locate } q] \leq 3 \cdot \sum_{i=1}^{n} \Pr(\Delta_i(q) \neq \Delta_{i-1}(q))$

$\leq 3 \cdot \sum_{i=1}^{n} \frac{4}{i} = 12 H_n = O(\log n)$