**Planar maps / Planar straight-line graphs**

Vertices = points in \( \mathbb{R}^2 \)
Edges = segments in \( \mathbb{R}^2 \)
Faces = components of \( \mathbb{R}^2 \setminus V \cup E \)

**Data structures for graphs:**

- **Adjacency / incidence list:**
  - List of neighbors of each vertex
  - Correspondence between edge records and darts

- **Doubly-connected edge list**
  - Represent edges \( e \rightarrow e \) as pairs of half-edges/darts \( e \rightarrow e \)

**Navigation:**

- \( \text{nxt}(d) \), \( \text{hnd}(d) \), \( \text{hprev}(d) \)
- \( \text{rev}(d) \)
- \( \text{next}(d) \), \( \text{prev}(d) \)

- Textbook uses a different explicit subset. Whatever.

Most of these are not represented explicitly.

E.g.

- \( \text{hnext}(d) = \text{rev}(\text{hnext}(d)) \)
- \( \text{nxt}(d) = \text{rev}(\text{nxt}(d)) \)
- \( \text{hrev}(d) \)

- \( \text{hnext} + \text{hprev} \) define a doubly-linked list of darts into each vertex

Vertices also store \( x, y \) coordinates

\( V, E, F \) store other attributes as necessary

Each vertex points into this list

Each face \( F \) points to some dart \( d \) with \( \text{left}(d) = F \)
If graph is disconnected, faces can have multiple boundary cycles.
Face keeps list of darts, one on each boundary cycle.

Maintenance:

\[
\begin{align*}
PQ & \rightarrow O_S \\
po & \rightarrow oq \\
\end{align*}
\]
Add vertex \( x \)\nAdd edges \( xp, x r, x g, x s \)
Delete edges \( pq, rs \)
(Deal with faces at end)

When sweep line gets here:

Split edges above & below (ptrs from BST)
Add new vertical edges (known can order at ends)
Deal with faces at the end
Infinite rays: sentinel vertices at \((0, \infty)\) and \((10, \infty)\)
Or large bounding box

Without coordinates, but with orderings:
combinatorial embedding
represents topological structure w/o geometry

Dual graph \( G^* \)

\[
\begin{align*}
\text{vertices} & = F^* \\
\text{edges} & = E^* \\
\text{faces} & = V^* \\
\end{align*}
\]

\[
\begin{align*}
h_{\text{next}} & = l_{\text{next}} = \text{rev}(h_{\text{next}}) \\
l_{\text{next}} & = h_{\text{next}} \text{ rev}^* = \text{rev} \\
\text{left-right reversed!} \\
\end{align*}
\]
Same data structure represents $G$ and $G^*$ (except for geometry)

**Fauler's Formula:** $V - E + F = 2$ (For connected map)

$V - E + F = 1 + K$  For map with $K$ components

Proof by contraction-deletion = tree-cotree