Given set $P$ of points called "sites"

Voronoi diagram = partition into regions:

$\text{Vor}(p) = \{ q \mid p \text{ is closest site to } q \}$

Perpendicular bisector

$$\| (a, b) - (x, y) \|^2 = \| (c, d) - (x, y) \|^2$$

$$(a-x)^2 + (b-y)^2 = (c-x)^2 + (d-y)^2$$

$$(a-c)x + (b-d)y = \frac{1}{2} (a^2 + b^2) - \frac{1}{2} (c^2 + d^2)$$

3 bisectors meet at the center of the circumcircle of the 3 sites

More than 3 sites:

$\text{Vor}(p)$ is intersection of $n-1$ halfplanes defined by perp. bisectors.

$\Rightarrow$ convex polygon

General position $\Rightarrow$ no 4 pts are cocircular $\Rightarrow$ every Vor. vertex has deg 3
Vor (p) is unbounded iff p is vertex of conv(P)
Unbounded Vor. edge \( \pm \) convex hull edge

Empty circles degenerate to empty halfplanes in the limit.

PSLG Euler's formula
\[ \Rightarrow \text{Vor diagram has } 2n - 2 - h \text{ vertices} \]
\[ \text{(not counting } \infty \text{) and } 3n - 3 - h \text{ edges} \]
(assuming general position)
otherwise \( \leq \)

Dual of Vor diagram = Delaunay triangulation (Delone)

Three point pqr define a Delaunay triangle iff circ(pqr) is empty
Delaunay edge pq \( \Leftrightarrow \)
A circle with pq inside other sites outside

PSLG \[ 2n - 2 - h \text{ faces} \]
\[ 3n - 3 - h \text{ edges} \]
where \( h = \# \text{convex hull edges} \)

Dual means same data structure
**Circles:**

Circle with center \((a,b)\) and radius \(r:\)

\[
(x-a)^2 + (y-b)^2 = r^2
\]

\[
x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0
\]

\[
x^2 + y^2 - 2ax - 2by + r^2 = 0
\]

Circle center \((a, b)\)

Radius \(\sqrt{r - (a^2 + b^2)}\)

**Circumcircle of** \((a,b), (c,d), (x,y)\):

\[
\det \begin{bmatrix}
1 & a & b \\
1 & c & d \\
1 & x & y \\
\end{bmatrix} = 0
\]

**Line thru** \((a,b), (c,d)\):

\[
\det \begin{bmatrix}
1 & a & b \\
1 & c & d \\
1 & x & y \\
\end{bmatrix} = 0
\]

Collinear \(\implies\) circle degenerates to a line

---

**Delannay triangulation of 4 points**

Equivalently:

\[
\begin{cases}
S \text{ is outside } \text{circ}(pqr) \\
Q \text{ is outside } \text{circ}(prs) \\
P \text{ is inside } \text{circ}(qrs) \\
R \text{ is inside } \text{circ}(pqs)
\end{cases}
\]

\[
\det \begin{bmatrix}
1 & px & py & px^2 + py^2 \\
1 & qx & qy & qx^2 + qy^2 \\
1 & rx & ry & rx^2 + ry^2 \\
1 & sx & sy & sx^2 + sy^2
\end{bmatrix} > 0
\]

(assuming \(pqr\) ccw)

More generally, a triangulation is locally Delannay if every interior edge passes this test.
Theorem: \( \text{locally Delaunay} = \text{Delaunay} \)

Lawson's flip algorithm

1. Triangulate \( P \) however locally
2. While there is a non-Delaunay edge Flip it

Theorem: This algo terminates after \( O(n^2) \) flips \( \Rightarrow \text{Delaunay } \Delta n! \)