**Voronoi Diagrams**

\[ \text{Vor}(p) = \{ q \in \mathbb{R}^2 \mid p \text{ is the closest site to } q \} \]

**Applications:**
- Nearest neighbor queries
- Interpolation

**Fortune's Algorithm - \(O(n \log n)\) time**

Voronoi edge = perp. bisector of \( pq \)

\[ \sqrt{(x-b)^2 + (y-d)^2} = \sqrt{(c-x)^2 + (d-y)^2} \]

\[ (a-c)x + (b-d)y = \frac{1}{2}(a^2 + b^2 - c^2 - d^2) \]

Voronoi vertex =
- Circumcenter of 3 sites
- Center of circumcircle

General position:
- No 3 pts collinear
- No 4 pts cocircular
More generally:

- Every Voronoi region is a convex polygon.
- Intersection of \( n-1 \) bisector half-planes.
- \( \text{Vor}(p) \) connected.
- \( \text{Compute} \ \text{Vor}(p) \) in \( O(n \log n) \) time.

- \( \text{Vor}(p) \) is unbounded iff \( p \) is a vertex of \( \text{conv}(P) \).

- Infinite Voronoi edges \( \iff \) convex hull edges.

Growing empty circle becomes empty halfplane in the limit.

Planar straight-line graph, every vertex has degree 3.

Euler's formula: \( n \) faces \( \Rightarrow \) 2n-2-h vertices, 3n-3-h edges where \( h = \# \text{hull vertices} \).

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**Delaunay Triangulation**

- Planar dual of Voronoi diagram.
- Triangulation of \( P \) (assuming gen. pos.)

- PSLG \( n \) vertices, 3n-3-h edges, 2n-2-h faces.
Standard DS for Voronoi = Standard DS for Delaunay

pq is a Delaunay edge iff some circle has p,q inside all other sites outside

pqr is a Delaunay triangle iff circumsphere (p,q,r) is empty

Circles:

\[(x-a)^2 + (y-b)^2 = r^2\]

\[x^2+y^2-2ax-2by + a^2+b^2-r^2 = 0\]

\[\text{Of } \text{coeff}(x^2) = \text{coeff}(y^2), \text{ coeff}(xy) = 0\]

\[x^2+y^2-2\alpha x - 2\beta y + \gamma = 0\]

Circle with center \((\alpha, \beta)\) radius \(\sqrt{\gamma - \alpha^2 - \beta^2}\)

Circumcircle of \((a,b), (c,d), (e,f)\):

\[
\det \begin{bmatrix}
1 & a & b & a^2+b^2 \\
1 & c & d & c^2+d^2 \\
1 & e & f & e^2+f^2 \\
1 & x & y & x^2+y^2
\end{bmatrix} = 0
\]

\[\text{degenerates to a line if } \text{pts are collinear}\]

If \(\det > 0\) and 3 pts are CCW
\((x,y)\) inside circle

4 points

\[p_q\text{ outside circ}(pqr)\]

\[
\begin{bmatrix}
4p_x & 4p_y & p_x^2 + p_y^2 \\
4q_x & 4q_y & q_x^2 + q_y^2 \\
4r_x & 4r_y & r_x^2 + r_y^2 \\
4s_x & 4s_y & s_x^2 + s_y^2
\end{bmatrix} > 0
\]

InCircle Test \((p,q,r,s)\)
A triangulation is locally Delaunay if every pair of adjacent \( \Delta \)s passes the incircle test.

"Every interior edge is locally Delaunay."

**Theorem:** Delaunay \( \Rightarrow \) locally Delaunay

*Proof on Thursday*

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**Lawson's Flip algorithm (1977)**

1. Triangulate \( P \)
2. Repeatedly flip bad edges until all edges are locally Delaunay

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**Theorem \( \Rightarrow \)**

*If* algo halts, final triangulation is Delaunay

- It does halt
- After \( O(n^2) \) flips
- Basis of \( O(n \log n) \)-time randomized incremental algorithm