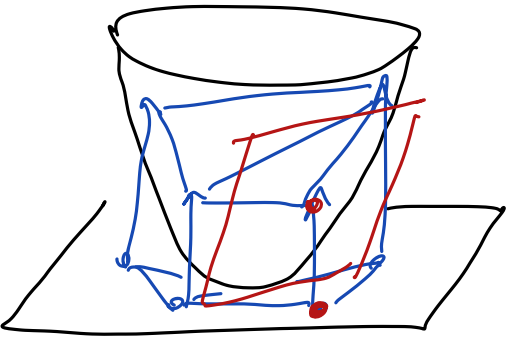


Lower convex hull of pts (empty halfspaces) $\xrightarrow{\text{projective}}$ upper envelope of planes

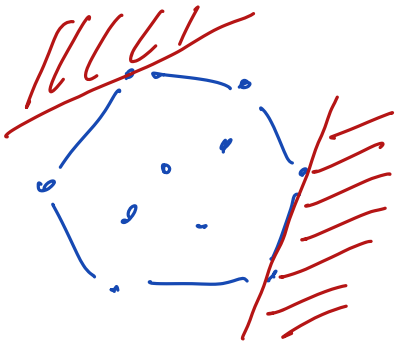
$\xleftarrow{\text{duals}}$

lift $\uparrow$ $\downarrow$ project

lift $\uparrow$ $\downarrow$ project

Delaunay triangulations (empty circles) $\xrightarrow{\text{dual}}$ Voronoi diagrams (nearest neighbors)

$\xleftarrow{\text{dual}}$



$(a,b) \xrightarrow{\text{lift}} (a, b, \frac{a^2+b^2}{2})$

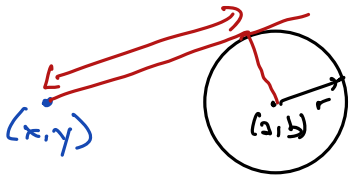
$(a,b,c)^* \xleftrightarrow{\text{dual}} z = ax + by - c$

$(a,b) \xrightarrow{\text{lift+dual}} z = ax + by - \frac{a^2+b^2}{2}$

tangent to $z = \frac{x^2+y^2}{2}$ at $(a, b, \longrightarrow)$

incircle test: $\det \begin{bmatrix} 1 & a & b & a^2+b^2 \\ 1 & c & d & c^2+d^2 \\ 1 & e & f & e^2+f^2 \\ 1 & g & h & g^2+h^2 \end{bmatrix} \longleftrightarrow$ orientation $\det \begin{bmatrix} 1 & a & b & \alpha \\ 1 & c & d & \beta \\ 1 & e & f & \gamma \\ 1 & g & h & \delta \end{bmatrix}$

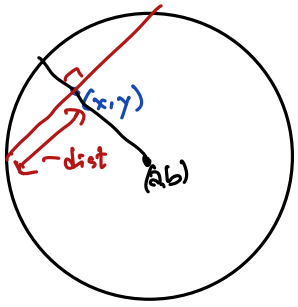
Power diagrams (weighted Vor. diagrams)

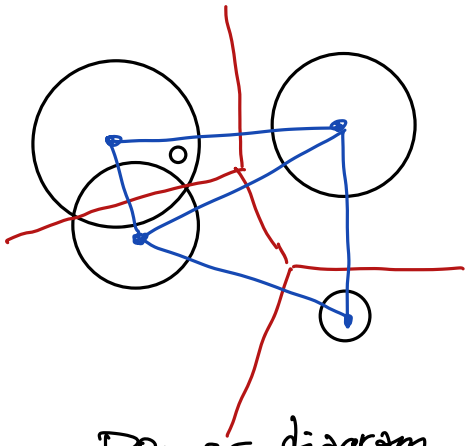


circle = weighted point

power distance from (x,y) to circle centered at (a,b) with radius r

is $(x-a)^2 + (y-b)^2 - r^2$

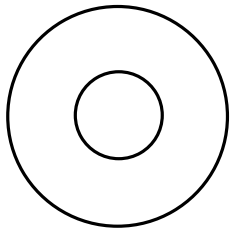




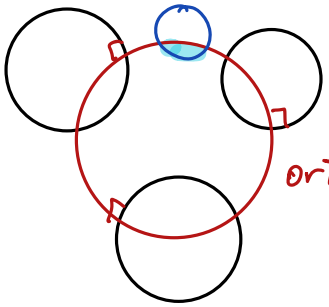
power diagram

Bisectors are lines

Some power regions may be empty

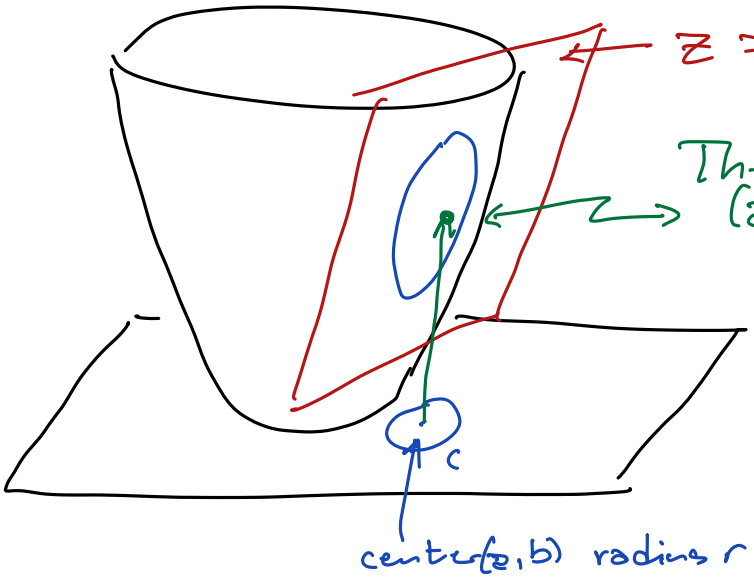


concentric: larger circle is always closer



ortho circle

$$\operatorname{sgn}\det\begin{bmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2-r_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2-r_2^2 \\ 1 & x_3 & y_3 & \vdots \\ 1 & x_4 & y_4 & \vdots \end{bmatrix}$$



$z = ax + by - \frac{1}{2}(a^2+b^2-r^2)$

The point $(a, b, \frac{1}{2}(a^2+b^2-r^2))$ can see inside lifted circle

weighted Vor diagram of circles = projection of upper envelope of lifted planes

dual

weighted Delaunay Δn of circles = projection of lower hull of lifted points

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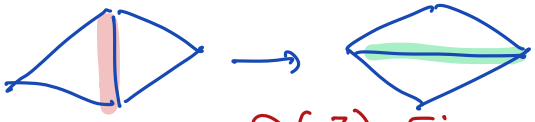
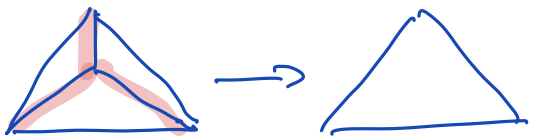
Flipping algorithm





$\Theta(n^2)$ flips worst case

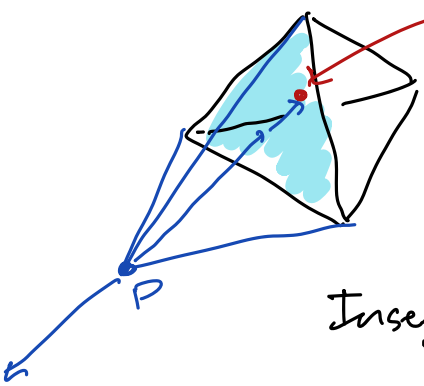
Randomized incremental:

test for local wt Delaunay before trisecting

$O(n^2)$ time/flips worst case

$O(n)$ exp # flips
$O(n \log n)$ exp time → 3D convex hulls!

RIC 3d hulls



Start with 4 pts → tetrahedron

Every pt remembers one "visible" facet (intersecting ray $\overrightarrow{op}$)

Insert(p):
- If p is "outside" its facet
- Connect p to facet edges
- Flip away nonconvex edges + vertices

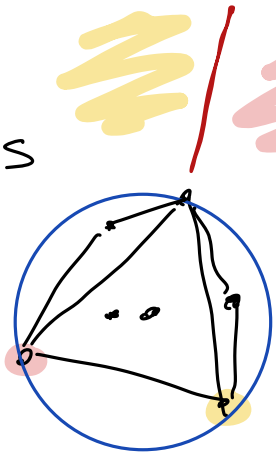
$O(n \log n)$ exp. time — same analysis

Anti-Delaunay triangulation

= projection of upper hull of lifted pts

= Δs characterized by full circle

Every other site inside circ(par)



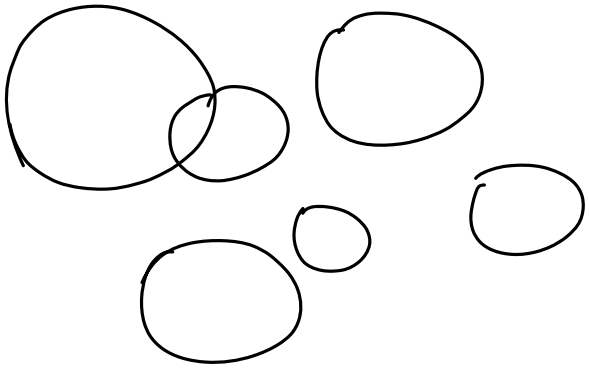
Δs of convex hull.

Anti-Voronoi diagram

= proj of lower envelope of lifted planes

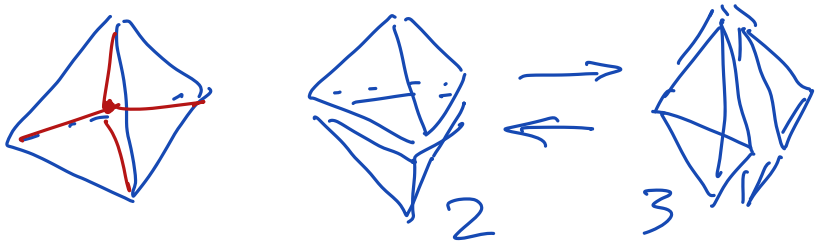
= partition by furthest neighbor

region is non empty ⟺ p is on conv hull



3d Del/Vor same ideas

$O(n^2)$ in worst case
$O(n)$ anti Del/Vor } complexity



output size

RIC via flips $\rightarrow$ $O(n \log n + k)$ time