Theorem: Locally Delaunay = Delaunay

Every pair of adjacent triangles passes the incircle test

\[ \text{S outside } \text{circ}(pqr) \]

Every triangle has an empty circumcircle

\[ \text{All points outside } \text{circ}(pqr) \]

Proof ($\Rightarrow$):

Let $T$ be any non-Delaunay triangulation of $P$

Choose $\triangle pqr$ and $z$ inside $\text{circ}(pqr)$

minimizing distance from $z$ to $pqr$

$\text{pr}$ separates $q$ from $z$

$\Rightarrow \text{pr}$ is not a convex hull edge

$\Rightarrow$ Another triangle $p'rs$

If $s=z \Rightarrow T$ is not locally Delaunay

Assume $s \neq z$

Let $D = \text{area between } \text{pr}^2$ and $\text{circ}(pqr)$ containing $z$

IF $s \in D \Rightarrow T$ is not loc. Del

Assume otherwise

$\text{circ}(prs)$ contains $D \Rightarrow$ contains $z$

$\text{dist}(z, prs) = \text{dist}(z, p^2) < \text{dist}(z, pr)$

contradicts choice of $pqr$ and $z$
Lawson’s Flip Algorithm
Start with any triangulation of $P$
Flip non-Del edges until none are left

Local-Global Delaunay $\Rightarrow$
- If this edge halts,
- Final triangulation is Delaunay

Linearization
- Circles in $\mathbb{R}^2$ $\rightarrow$ planes in $\mathbb{R}^3$
- [Voronoi 1906, Maxwell 1864]

Paraboloid lifting map $\pi_t(x, y) = (x, y, \frac{1}{2}(x^2+y^2))$

$z = \frac{1}{2}(x^2+y^2)$
$z = ax + by - c$

$s$ is outside $\text{circ}(pqr)$ $\iff$ $\pi(s)$ is above plane $(\pi(p), \pi(q), \pi(r))$
$s$ above plane $(\beta, \delta, \gamma)$

Incircle test
$\det \begin{bmatrix} 1 & x_1 & y_1 & \frac{1}{2}(x_1^2+y_1^2) \\ 1 & x_2 & y_2 & \frac{1}{2}(x_2^2+y_2^2) \\ 1 & x_3 & y_3 & \frac{1}{2}(x_3^2+y_3^2) \\ 1 & x_4 & y_4 & \frac{1}{2}(x_4^2+y_4^2) \end{bmatrix}$

3d orientation test
$\det \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}$
det = 6 \cdot \text{vol}

\( p_r \) is locally Delaunay

\( \hat{p}_r \) is locally convex

\( \hat{q} \) is above \( \hat{p}_r \)

\( p_r \) is not locally Delaunay

\( \hat{p}_r \) is locally concave

\( \hat{q} \) is above \( \hat{p}_r \)

Delaunay triangulation of \( P \) is projection of lower convex hull of \( \hat{P} \)

\textbf{Lawson's algorithm}

Lift any triangulation \( T \) to paraboloid \( \rightarrow \hat{T} \)

Flip any concave edges until none are left

Every flip lowers the surface \( \hat{T} \)

\( \hat{p}_r \) is above \( \hat{T} \) after the flip forever
Done after $O(n^2)$ flips

Randomized incremental algorithm
Sibson, Green 77
Guibas, Knuth, Sharir 90
history dag: Seidel

Insert points in random order
maintain Delaunay triangulation

For each point $p \in P$
- Find $\Delta$ containing $p$ in current triangulation
- trisect $\Delta$ at $p$
- Lawson: flip away non-Del edges

Every flip removes
an edge opposite $p$
adds an edge incident to $p$

$$\#\text{flips} = \deg(p) - 3$$

$p$ is random $\Rightarrow E[\deg(p)] < 6$ $\Rightarrow E[\#\text{flips}] < 3$

How do we start?

Sentinel Triangle