Variants of Voronoi/Delaunay

Paraboloid lifting map
\[ p = (a, b) \mapsto (a, b, \frac{a^2 + b^2}{\epsilon}) = \hat{p} \]
Delaunay triangulation of \( P = \text{proj of lower hull of } \hat{P} \)

Planar map duality
\[ \text{Delaunay}(P) \leftrightarrow \text{Voronoi}(P) \]

Paraboloid isn't special

Stereographic lifting

Sphere is a projective transformation of paraboloid.
Power diagrams (weighted Voronoi diagram)

Weighted points \( = \) circles in \( \mathbb{R}^2 \)

Power distance from pt \((x,y)\) to circle \((a,b),r\) is
\[
    d^2 = (x-a)^2 + (y-b)^2 - r^2
\]
outside \(>0\)
on \(=0\)
inside \(<0\)

Power diagram of a set of circles partitions plane by min squared power distance.

Some circles can have empty power regions

Dual = weighted Delaunay triangulation

Weighted incircle test

weighted Del = locally wt-Del.

lifting map
\[
    ((a,b),r) \rightarrow (a, b, \frac{1}{2}(a^2 + b^2 - r^2))
\]
wt Del = projection of lower hull
\[
    ((a,b),r) \rightarrow z = ax + by - \frac{1}{2}(a^2 + b^2 - r^2)
\]
wt Vor = projection of upper envelope

Reverse projection
\[
    (a, b, c) \rightarrow (a, b, \sqrt{a^2 + b^2 - 2c})
\]
Flipping algorithm $\rightarrow$ Randomized incr. algorithm

**two minor changes**

- Test local Delaunay condition before trisecting
- If weighted non-Delaunay edge separates triangles with non-convex union, delete a vertex

Worst case: $O(n^2)$ flips
Expected: $O(n \log n)$ time

Lower convex hulls in $\mathbb{R}^3$!

**Upper convex hull? $\rightarrow$ Flip vertically $z \rightarrow -z$**

- Compute lower hull

Anti-Delaunay triangulation $\rightarrow$ Anti-Voronoi diagram

Circumcircle of every $A$ contains every site
Locally anti-Del = anti-Del

Same flipping algorithm
Same analysis

Convex hulls in $\mathbb{R}^3$

$O(n \log n)$ exp. time
Convex hulls in $\mathbb{R}^d$

$O(n \log n + n^{d/2})$ exp. time

↑ worst-case output size