Paper Presentations
registration form Apr 15

Smallest annulus
\[ \rightarrow \text{LP with 4 variables } \]
\[ a, b, r^2-a^2-b^2, r^2-a^2-b^2 \]

Smallest enclosing disk
Sylvester [1852]

center \((a, b)\) and radius \(r\)

\[ \min r^2 \]
\[ \text{s.t. } (x_i-a)^2+(y_i-b)^2 \leq r^2 \text{ for all } i \]

Smallest annulus always has 4 points on boundary

But smallest disk has either 2 or 3 pts on bdry

even assuming general position
\[ \Rightarrow \text{ NOT LP } \]

But it's "LP enough"

Seidel's algorithm exploits combinatorial properties of LP:

1. Uniqueness (assuming g.p.)
2. Uniquely determined by tight constraints.
3. Exchange property: If \( \text{OPT}(H-h, B) \neq h \) Then \( \text{OPT}(H, B) = \text{OPT}(H-h, B+h) \)
Smallest disk has these properties

Uniqueness & convexity:

\[ P \subseteq D(c, r) \Rightarrow c \in \bigcap_{P \in P} D(p, r) \]

Some min radius where \( \bigcap D \) is non-empty:

\[ \bigcap D = \text{point} \]

Lemma: Let \( P \) and \( T \) be disjoint point sets in \( \mathbb{R}^2 \)

\[ \text{Min} D(P, T) = \text{smallest disk } D \text{ with } P \subseteq D \]

If there is a disk \( D \) with \( P \subseteq D \) and \( T \cap \partial D \)

the smallest such disk is unique.

Proof:

Let \( D_1 \) and \( D_2 \) be equal-radius disks with \( P \) inside and \( T \) on boundary

\[ P \subseteq D_1 \cap D_2 \Rightarrow D_1 \cap D_2 \neq \emptyset \]

\[ T \subseteq \partial D_1 \cap \partial D_2 \]

Let \( c' = c_1 c_2 \cap r_1 r_2 \) be midpoint of \( r_1, r_2 \)

\[ D' = \text{disk centered at } c' \text{ with } r_1, r_2 \text{ on boundary} \]

\[ (1) \text{ radius } (D') < \text{ radius } (D_1) = \text{ radius } (D_2) \]

\[ (2) P \subseteq D_1 \cap D_2 \subseteq D' \Rightarrow D_1 \text{ and } D_2 \text{ not smallest enclosing disks} \]
Pivoting Lemma: Let \( P, R \) be disjoint sets.
For any \( p \in P \),
1. \( p \in \text{MinD}(P-p, R) \Rightarrow \text{MinD}(P, R) = \text{MinD}(P-p, R) \)
2. \( p \notin \text{MinD}(P-p, R) \Rightarrow \text{MinD}(P, R) = \text{MinD}(P-p, R+p) \)

**Proof:**

1. Suppose \( p \in D \) \hspace{1cm} \( \text{MinD}(P-p, R) \)
   
   If smaller disk \( D' \) contained \( P \) and \( R \) on bdry
   
   then \( D' \) would contain \( P-p \) and \( R \) on bdry
   
   contradicting def.: \( D \).

2. Suppose \( p \notin D \) \hspace{1cm} \( D' = \text{MinD}(P, R) \)

   Move center \( c_t \) along ray \( \overrightarrow{ct} \)

   \( D_t = \) disk centered at \( c_t \)
   
   that has \( r_0, r_1 \) on bdry

   Radius of \( D_t \) must increase monotonically with \( t \)

   \( r_t^2 = r_0^2 + t^2 \)

   \( D' \) is the first disk \( D_t \) with \( p \in D_t \equiv p \notin \text{MinD} \)

Welzl’s MiniDisk algorithm:

\[ \text{MinD}(P, R) : \]

\[
\begin{cases}
\text{if} \left| \mathbb{R} \right| > d + 1 \\
\text{return INFEASIBLE}
\end{cases}
\]

\[
\begin{align*}
\text{else if } & P = \emptyset \\
\text{compute } & \text{MinD by brute force}
\end{align*}
\]

\[
\begin{align*}
\text{else} \\
& p \leftarrow \text{random point in } P \\
D & \leftarrow \text{MinD}(P-p, R) \\
\text{if } & p \in D \\
& \text{return } D \\
\text{else} \\
& \text{return } \text{MinD}(P-p, R+p)
\end{align*}
\]
Try running this when \( |P|=3, \ R=\emptyset, \)

Run in \( O(n) \) expected time by Seidel's analysis

\( O((d+1)!n) \)

\[ \Downarrow \]

\( O(d\cdot d!\cdot n) \)

Similar algs for other "LP-type" problems

Monotonicity: \( OPT(A) \leq OPT(A+x) \)

Locality: \( A \subseteq B \quad OPT(A) = OPT(B) = OPT(A+x) \quad \Rightarrow \quad = OPT(B+x) \)

Quasi-convex programming

\( f_1, f_2, \ldots, f_n \) convex sublevel sets

\[
\text{Find } \min \left( \max f_i \right)
\]

\( F^{-1}(-\infty, t) \)

On the other hand if \( d \) is large

Find \( \min D \) using a variant of simplex