Paper presentations — last week April 20 min each
Register on Gradescope by April 15

Applications of Line Arrangements
(via duality)

(Half plane) Discrepancy

\( P = n \) points in \( \mathbb{R}^d \)

For any halfplane \( h \) define
\[
\mu(h) = \text{area} (h \cap \mathbb{R}^d)
\]
\[
\mu_p(h) = |P \cap h| / n
\]

Discrepancy
\[
\Delta(h) = |\mu(h) - \mu_p(h)|
\]

Given \( P \) compute
\[
\Delta(P) = \max_h \Delta(h)
\]

0(\( n \)) lines of this form
(0(1) per point)
0(\( n^2 \)) time by brute force

Worst halfspace is either bounded by

- line thru two points in \( P \)
- line thru one point in \( P \) at midpoint of \( \ell \cap \mathbb{R}^d \)

We can compute \( \mu(h) \) for any \( h \) in \( O(1) \) time

Bottleneck: compute \( \mu_p(h) \) for all candidates \( h \).

For every pair of points, find # points below that line
For every pair of lines \( p^*, q^* \), find the number of lines above point \( p^* \) or \( q^* \).

Build surgh \((P^*)\) 

1. Trace that each level in \( O(1) \) time per vertex.

- or -

2. Compute level of any vertex, WFS

\( \Theta(n^2) \) time

**Ham Sandwich Cuts**

Given two point sets \( P \) and \( B \), find a line that bisects both sets.

If we rotate points, maintain vertical bisectors, \( IVT \) at some point lines coincide.
Build arrangements of $R^t$ and $B^t \leq O(n^2)$

Trace median levels in both angles \(\leq O(n^2)\)

IVT $\Rightarrow$ median levels intersect

\[
\text{We don't know worst case complexity of median level!}
\]

\[
\begin{align*}
n \cdot 2^{2^{\Omega(\log n)}} & \quad O(n^{4/3})
\end{align*}
\]

Minimum area triangle

Given set $P$ find $pq, r \in P$ minimizing area$(\Delta pqr)$

Brute force: $O(n^3)$

Fix two points $p$ and $q$

Min area $\Delta pqr \Leftrightarrow$

\[
\text{dist}(r, pq) \text{ is minimized}
\]

\[
\text{dualize}
\]

\[
\text{Area} = \frac{1}{2} b h
\]
For each vertex in argh($P^*$)
we want closest line above or below

we can build argh + trap decom
in $O(n^3)$ time

Do any three points lie on a line?
Do any three lines pass thru a common point?

3SUM: Given a set $X$ of $n$ numbers
do any three elements of $X$ sum to 0?

$O(n^2)$ time

Define $X = \{(x, x^3) \mid x \in X\}$

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c)$$

Matching $\Omega(n^2)$ lower bound in limited model

Grønlund Pettie 2014: $O(n^2 \log^4 n / \log n)$

Chan 2018: $O(n^2 \log^6 \log n / \log^2 n)$

Fastest known

3SUM conjecture: $O(n^{2-\epsilon})$ time is impossible for all $\epsilon > 0$
even for integers between $n^3$ and $n^3$.\n