Applications of line arrangements

Let $P$ be a set of $n$ points in the unit square $[0,1]^2$.

For any half-plane $h$, define

$$\mu(h) = \text{area of } h \cap [0,1]^2$$

$$\mu_P(h) = \frac{|P \cap h|}{|P|}$$

discrepancy $\Delta_P(h) = |\mu(h) - \mu_P(h)|$

discrepancy of $P = \sup_h \Delta_P(h)$

Motivated by ray-tracing/anti-aliasing, Monte Carlo integration

Continuity arguments imply line $l$ with max discrepancy has one of two forms:

- Through one point $p \in P$ at midpoint of $l \cap [0,1]^2$

  Only $O(n)$ of these ($\leq 8$ per point in $P$)

- Through two points in $P$

  But $O(n^2)$ of these

  We can compute $\mu(h)$ in $O(1)$ time

  But what about $\mu_P(h)$?

Given set $P$ of $n$ points in $\mathbb{R}^2$,

Find # points in $P$ above each line thru two points in $P$

Duality FTW!
Compute the level of every vertex of $\text{arrgh}(L)$

**Build $\text{arrgh}(L)$:**

- Compute level of any vertex by brute force
- Compute remaining levels by WFS

Alternatively, compute left "end" of each level by sorting slopes
- Walk along each level in $O(1)$ time per vertex

**Ham-Sandwich Cuts**

- Given two sets of points $R$ and $B$ in the plane, find a line that bisects both sets

$(\text{Ham-sandwich theorem: There is a plane that cuts both slices of bread and the ham exactly in half even if one slice of bread is on your head and the other one is the moon.})$
Existence proof (2D):
Assume both sets have odd # points

Let \( l_B \) and \( l_Z \) be unique vertical lines that bisect \( B \) and \( \overline{R} \)

WLOG \( l_B \) is left of \( l_Z \).

Continuously rotate the plane.

The **vertical** bisecting lines change continuously!

After half turn, 
\( l_B \) is right of \( l_Z \).

So at some angle, they must coincide! \( \square \)

**Algorithm:**
Keep points fixed, vary slopes of bisectors from \(-\infty\) to \(\infty\)

In the dual:
Two families of lines \( B^* \) and \( R^* \)

move points \( l_B^* \) and \( l_Z^* \) from left to right along median levels of \( \text{arrgh}(B^*) \) and \( \text{arrgh}(R^*) \)

These levels must intersect!

Construct arrangements of \( B^* \) and \( R^* \)

Walk along median levels to find intersection

\( O(n^2) \) time

\( O(n^{4/3}) \) \( \cdot \binom{n/2}{r} \), (ignore)

we don’t actually know how long this takes!!
Minimum-area triangles:

Given a set \( P \) of \( n \) points, find three points in \( P \) spanning minimum (unsigned!) area.

Naïve: \( O(n^3) \) time

Fix two points \( p \) and \( q \)
For any point \( r \), we have

\[
\text{area}(pqr) = \frac{1}{2} b \cdot h
\]

where

\[
b = |p.x - q.x| \quad \text{and} \\
h = \text{vertical distance from } r \text{ to } pq
\]

So min area triangle \( \Delta pqr \) uses 3rd point closest to \( pq \)

In the dual:

For each vertex in \( \text{arrgh}(P^*) \), we want closest line above or below

Trapezoidal decomposition!

Build \( \text{arrgh}(P^*) \)
Build \( \text{trap. decomp} \)
Find shortest vertical edge at each arrangement vertex \( \Rightarrow O(n^2) \) time
Can we do better? **Nobody knows!**

**Related problem:** **3SUM**: Given $n$ numbers, do any 3 sum to 0?

**Easy $O(n^2)$-time algorithm**

For any set $X$, let $X = \{x, x^2, x^3\}$ \( x \in X \)

Three elements of $X$ sum to 0 iff

Three points in $X$ are collinear!

\[ \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c)(b-a)(c-b)(c-a) \]

**Proof:**

**Matching $\Omega(n^2)$ lower bound in weak model of computation.**

**First subquadratic algo for 3SUM:** (without bit tricks)

- Granlund Pettie 2014: $O(n^2 \frac{\log \log n}{\log n})$ expected
- Gold Sharir 2017: $O(n^2 \frac{\log \log n}{\log n})$
- Chan 2018: $O(n^2 \frac{\log \log n}{\log^2 n})$

**3SUM conjecture:** $O(n^{2-\delta})$ time is impossible for all $\delta > 0$
even for integers between $-n^3$ and $n^3$

**But no $o(n^2)$-time algorithm for detecting collinear triples of points!**