1. Recall that a simple closed curve is ***polygonal*** if its image is the union of a finite number of line segments. A ***polygon*** is the closure of the interior of a simple closed polygonal curve. The boundary of a polygon P is denoted $\boldsymbol{\partial P}$.

   (a) Let G_n be a regular n-gon centered at the origin, with one vertex at $(1,0)$. Describe a homeomorphism $\phi_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi_n(\partial G_n) = S^1$.

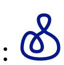
   (b) Describe an algorithm for the following problem: Given a simple polygon P, construct a homeomorphism $\phi \colon P \to G_n$ such that $\phi(P) = G_n$. The input polygon P is represented as an array of n vertices in (say) counterclockwise order. *[Hint: Any polygon with holes can be triangulated in $O(n \log n)$ time. How do you want to represent the output homeomorphism?]*

   (c) Describe an algorithm to construct a homeomorphism $\phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(P) = G_n$. Together with part (a), this proves the Jordan-Schönflies theorem for polygons.

2. A ***cycle*** in a topological space X is a continuous map $\gamma \colon S^1 \to X$. A ***(free) homotopy*** between cycles γ and δ in X is a continuous function $h \colon [0,1] \times S^1 \to X$, such that $h(0,\theta) = \gamma(\theta)$ and $h(1,\theta) = \delta(\theta)$ for all $\theta \in S^1$. Two cycles are ***freely homotopic*** if there is a free homotopy between them.

   (a) Loops and cycles are almost identical—any cycle can be turned into a loop by choosing a basepoint, and any loop can be transformed into a cycle by ignoring the basepoint. Describe a pair of closed curves in a polygon with holes that are freely homotopic *as cycles*, but not path-homotopic *as loops*. (Recall that a path homotopy keeps the basepoint of the loop fixed.)

   (b) Given a triangulated polygon P with holes in the plane and a polygonal cycle γ in P, describe an algorithm to compute the shortest cycle γ^* that is freely homotopic to γ. Don't develop the algorithm from scratch; just describe the necessary changes to Hershberger and Snoeyink's algorithm to compute shortest homotopic paths.
   [Hint: Prove that γ^* passes through a vertex of P.]

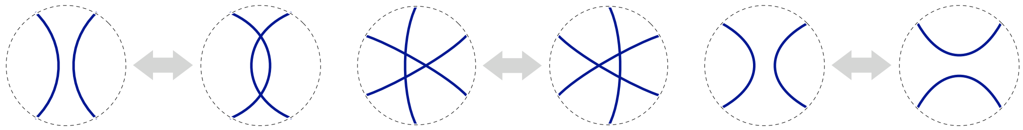
3. (a) Describe a sequence of Whitney moves that transforms the following curve into a circle: 

   (b) Suppose you are given two normal curves γ and δ, each of which have at most n points of self-intersection. Describe an algorithm to transform γ into δ using a sequence of Whitney moves. (In other words, give an algorithmic proof of the Whitney-Graustein theorem for normal curves.)

   (c) How many moves does your algorithm require in the worst case, as a function of n?

4. A ***homobathy***[1] between two normal closed curves as a sequence of Whitney moves and ***exchanges***, illustrated on the next page. An exchange move may split a normal curve into two curves, or merge two normal curves into one; thus, a homobathy is *not* a special type of homotopy. We say that two normal curves are ***homobathic*** if there is a regular homobathy between them. The ***length*** of a homobathy is the number of Whitney moves and exchanges.

[1]From the Greek $o\mu o\varsigma$ (homo- 'same') + $\beta\alpha\theta\upsilon\varsigma$ (bathys 'deep'), by analogy to 'immersion' (= Latin im- + mergere 'dip/plunge'). I couldn't find any standard terminology, so I just made something up; alternate suggestions are welcome!



In all three cases, nothing changes outside the circle during the move.

(a) Prove that two normal curves on the sphere are homobathic if and only if they are regularly homotopic.

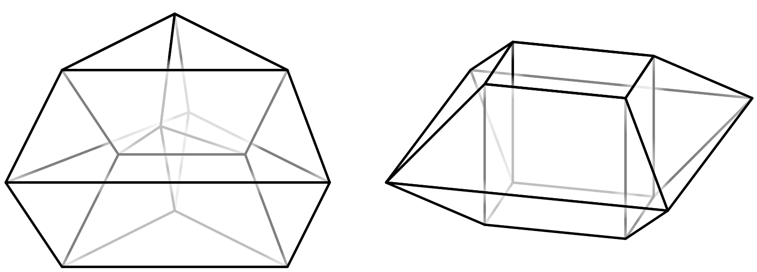
(b) Let C be a set of regular curves on the sphere with n self-intersection points. Prove that if n is even, there is a homobathy of length $O(n)$ transforming C into a single circle.

(c) Sketch a proof that for any even integer n, any quadrilateral mesh of the sphere with n quads can be extended to a topological hex mesh of the ball with $O(n)$ hexes. Don't reinvent the wheel; just describe the necessary changes to Mitchell and Thurston's construction.

Eppstein [3] provides a solution to part (c) using different techniques, but I think this approach is more straightforward.

★5. Which quad meshes on the *torus* can be extended to (topological) hex meshes of its interior?

This question is considerably more subtle than the corresponding question for the sphere, because the existence of a compatible hex mesh depends on how the quad mesh (or equivalently, the torus) is embedded in $\mathbb{R}^3$. Consider the two toroidal meshes shown below, both obtained by identifying opposite sides of a 3×4 grid. The mesh on the left is clearly the boundary of three hexes. On the other hand, Mitchell [4] proved that in any hex mesh, any cycle of boundary edges that bounds an interior disk must have even length; this implies that the mesh on the right cannot be extended to a hex mesh. See also Eppstein [3] and Bern and Eppstein [1].



Isomorphic quad meshes of the torus. Only the first can be extended to an interior hex mesh.

References

[1] M. Bern and D. Eppstein. Flipping cubical meshes. *Proc. 10th Int. Meshing Roundtable*, 19–29, 2001. ⟨http://www.andrew.cmu.edu/user/sowen/abstracts/Be812.html⟩. Preliminary version of [2].

[2] M. Bern, D. Eppstein, and J. Erickson. Flipping cubical meshes. *Engineering with Computers* 18:173–187, 2002. Full version of [1].

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[4] S. A. Mitchell. A characterization of the quadrilateral meshes of a surface which admit a compatible hexahedral mesh of the enclosed volume. *Proc. 13th Ann. Symp. Theoret. Aspects Comput. Sci.*, 456–476, 1996. Lecture Notes Comput. Sci. 1046, Springer-Verlag.