1. Consider a polygonal schema Π with a single face and n edges. Let Λ be the set of edge labels, and let $\bar{\Lambda} = \{\bar{x} \mid x \in \Lambda\}$. The *signature* of Π is a word in $(\Lambda \cup \bar{\Lambda})^*$ describing the sequence of edges on its single face; each edge label appears exactly twice (possibly barred). The signature completely determines the homeomorphism type of the 2-manifold $\Sigma(\Pi)$.

   Let $w_1, w_2, \ldots, w_k$ be words in Λ^*, such that each symbol in Λ appears exactly once in exactly one w_i. Let w^R denote the reversal of any word w, and let $\overline{w}$ denote the 'inverse' of w, obtained by barring each letter in w^R. Thus, if $w = \mathtt{abc}$, then $w^R = \mathtt{cba}$, $\overline{w} = \overline{\mathtt{cba}}$, and $\overline{w}^R = \overline{\mathtt{abc}}$.

   (a) Which 2-manifold has a polygonal schema with signature $w_1\, w_1^R\, w_2\, w_2^R \cdots w_k\, w_k^R$?

   (b) Which 2-manifold has a polygonal schema with signature $\overline{w_1}\, w_1^R\, \overline{w_2}\, w_2^R \cdots \overline{w_k}\, w_k^R$?

   Prove your answers are correct. For example, $\mathtt{abccbadeed}$ is an example for part (a), and $\overline{\mathtt{cba}}\mathtt{cba}\overline{\mathtt{de}}\mathtt{ed}$ is an example for part (b), where $w_1 = \mathtt{abc}$ and $w_2 = \mathtt{de}$.

2. Euler's formula relates the number of vertices, edges, and faces in a combinatorial surface to its Euler genus: $V - E + F = 2 - \bar{g}$. (Recall that $\bar{g} = 2g$ if the surface is orientable and $\bar{g} = g$ otherwise.)

   (a) A ***triangulation*** is an embedded graph in which every facial walk has length 3. (For example, this is the simplest triangulation of the sphere:  .) Let T be a triangulation with n vertices of a surface of Euler genus $\bar{g}$. Exactly how many edges and triangles does T have?

   (b) Find absolute constants α, β, and γ (not depending on n or $\bar{g}$) with the following property:

   > For all non-negative integers n and $\bar{g}$ such that $n \geq \gamma\bar{g}$, every n-vertex graph embedded on a surface of Euler genus $\bar{g}$ has an independent set of size n/α, in which every vertex has degree at most β.

   Recall that an *independent set* in a graph G is a subset of the vertices of G, no two of which are connected by an edge in G.

   (c) Describe an algorithm to find an independent set of low-degree vertices (as described in part (b)) in a given surface graph in $O(n)$ time (with no hidden dependence on $\bar{g}$).

   (d) Describe an algorithm to compute the minimum spanning tree of an n-vertex graph embedded on a surface of Euler genus $\bar{g}$ in $O(n + \bar{g}\log\bar{g})$ time. *[Hint: Modify Borůvka's algorithm.]*

3. Let G be a cellularly embedded graph on a surface Σ *with boundary*. A ***cut graph*** is subgraph H of G such that the closure of $\Sigma \setminus H$ is a disk. A ***retract*** is a subgraph H of G such that every component of $\Sigma \setminus H$ is an *annulus* with one boundary in H. A cut graph (or retract) is minimal if no proper subgraph is a cut graph (or retract, repsectively). For example, a minimal cut graph of an annulus is a path from one boundary to the other, and a minimal retract of an annulus is a cycle homotopic to the boundaries.

   A ***pair of pants*** is a sphere minus three open disks. In the problems below, let G be a graph with non-negatively weighted edges, cellularly embedded in a pair of pants Σ.

   (a) Describe an algorithm to find the minimum-length cut graph in G in $O(n \log n)$ time. *[Hint: What does a minimal cut graph of a pair of pants look like?]*

   *(b) Describe an efficient algorithm to find the minimum-length retract in G. *[Hint: What does a minimal retract of a pair of pants look like?]*

4. (a) Let Σ be a combinatorial 2-manifold, where each corner x of each face of Σ is assigned a positive real number $\angle x$, called the ***angle*** at x. Let corners(v) or corners(f) denote the set of corners incident to a vertex v or a face f, respectively. We define the ***curvature*** of each vertex and face as follows:[1]

$$\kappa(v) := 1 - \sum_{x \in \text{corners}(v)} \angle x \qquad\qquad \kappa(f) := 1 - \sum_{x \in \text{corners}(f)} \big(1/2 - \angle x\big).$$

Prove the ***combinatorial Gauß-Bonnet theorem*** [5, 6, 3]:

$$\sum_{\text{vertex } v \text{ of } \Sigma} \kappa(v) + \sum_{\text{face } f \text{ of } \Sigma} \kappa(f) = \chi(\Sigma)$$

(b) Suppose every face of Σ is a triangle. Prove the following special case of the combinatorial Gauß-Bonnet theorem [1]:

$$\sum_{\text{vertex } v \text{ of } \Sigma} (6 - \#\,\text{corners}(v)) = 6\chi(\Sigma).$$

(c) Now suppose Σ is a *subcomplex* of some combinatorial 2-manifold. Let $\chi(v)$ denote the number of edges incident to v, counting loops twice, minus the number of corners incident to v.[2] We now redefine the curvature of a vertex v as follows:

$$\kappa(v) := 1 - \frac{\chi(v)}{2} - \sum_{x \in \text{corners}(v)} \angle x.$$

Prove that the combinatorial Gauß-Bonnet theorem holds in this more general setting [2, 7].

(d) Suppose the surface Σ' is homeomorphic to a disk, and every face and *interior* vertex of Σ' has curvature at most 0. Prove that at least three boundary vertices of Σ' have strictly positive curvature.

(e) Finally, let T be a tiling of the hyperbolic plane by n-gons, meeting at vertices of degree n, for some integer $n > 4$. Let C be a non-trivial simple cycle carried by the edges of this tiling. Prove that one of the following situations must occur:

- C is the boundary of a face of T.
- C contains at least $n-1$ consecutive edges of at least two faces of T.
- C contains at least $n-2$ consecutive edges of at least three faces of T.

[Hint: Consider the dual tiling. Choose the right angles.]

[1] These definitions use the *circle* as the unit of angular measurement.

[2] $\chi(v)$ is the Euler characteristic of the *link* of x. In fact, this generalization holds for *any* complex.

★5. Let G be a *directed* graph that is cellularly embedded on an orientable 2-manifold Σ (possibly) with boundary. Describe a polynomial-time algorithm to compute the shortest *directed* cycle in G that is non-contractible in Σ. (As far as I know, the only nontrivial surface for which a solution is known is the annulus [4].)

$\aleph_0$. The ***English homophony group*** EH? is the group generated by the letters of the English alphabet (A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z), where any two English words with the same pronunciation define the same group element. Find a group presentation for EH? with as few generators and relators as possible. Prove that your presentation is correct and minimal.

For example, because the words BEAR and BARE are homophones, the word $\mathtt{BEAR}\overline{\mathtt{ERAB}}$ represents the identity element in EH?. The words EAR and ARE also represent the same element of EH?, even though they are *not* homophones, because $\mathtt{EAR} = \overline{\mathtt{B}}\mathtt{BEAR} = \overline{\mathtt{B}}\mathtt{BARE} = \mathtt{ARE}$. Similarly, because DESERT and DESSERT are homophones, the letter S actually represents the identity element.

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