Arithmétique ! algèbre ! géométrie ! trinité grandiose ! triangle lumineux ! Celui qui ne vous a pas connues est un insensé ! Il mériterait l'épreuve des plus grands supplices ; car, il ya du mépris aveugle dans son insouciance ignorante. . . .

[Arithmetic! Algebra! Geometry! Grandiose trinity! Luminous triangle! Whoever has not known you is a fool! He deserves the most intense torture, for there is blind contempt in his ignorant indifference. . . .]

— Le Comte de Lautréamont [Isidore Lucien Ducasse], Chant Deuxième, *Les Chants de Maldoror* (1869)

À bas Euclide ! Mort aux triangles ! *[Down with Euclid! Death to triangles!]*

— Jean Dieudonné, keynote address at the Royaumont Seminar (1959)

15 Examples of Cell Complexes

15.1 Proximity Complexes of Point Clouds

Point clouds are an increasingly common representation for complex geometric objects or domains. For many applications, instead of storing an explicit description of the domain, either because the object is too complex or because it is simply unknown, it may be sufficient to store a representative sample of points from the object. Typical sources of point-could data are scanners (such as digital cameras, laser range-finders, LIDAR, medical imaging systems, and telescopes), edge- and feature-detection algorithms from computer vision, locations of sensors and ad-hoc network devices, Monte Carlo sampling and integration algorithms, and training data for machine learning systems.

By themselves, point clouds have no interesting topology. However, there are several natural ways to impose topological structure onto a point cloud, intuitively by 'connecting' points that are sufficiently 'close'. If the underlying domain is sufficiently 'nice' and the point sample is sufficiently 'dense', we can recover important topological features of the underlying domain.

15.1.1 Aleksandrov-Čech Complexes: Nerves and Unions

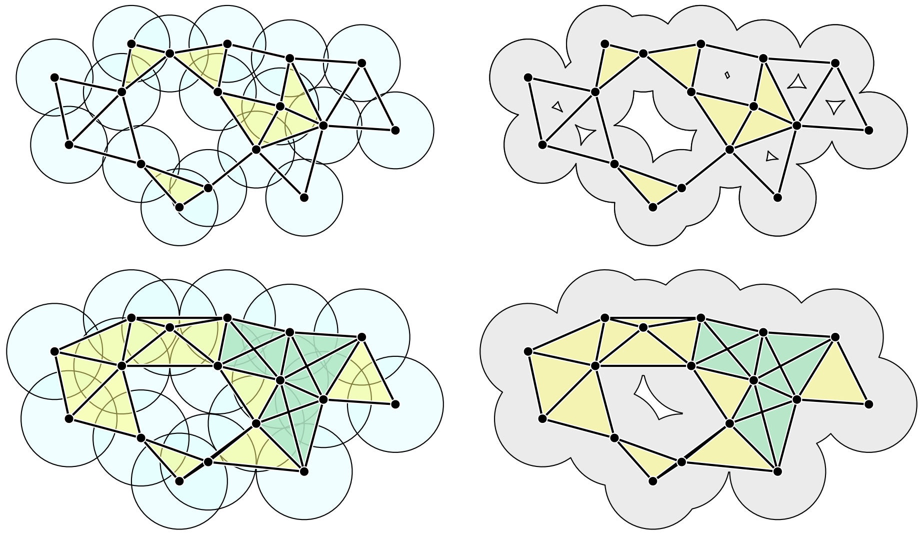
Let P be a set of points in some metric space S, and let ε be a positive real number. Typically, but not universally, the point set P is finite and the underlying space S is the Euclidean space $\mathbb{R}^d$. The ***Aleksandrov-Čech complex*** $\boldsymbol{A\check{C}_\varepsilon(P)}$ is the ***intersection complex*** or ***nerve*** of the set of balls of radius ε centered at points in P. That is, $k+1$ points in P define a k-simplex in $A\check{C}_\varepsilon(P)$ if and only if the ε-balls centered at those points have a non-empty common intersection, or equivalently, if those points lie inside a ball of radius ε. Formally, the Aleksandrov-Čech complex is an ***abstract*** simplicial complex; its simplices can overlap arbitrarily and can have arbitrarily high dimension. Aleksandrov-Čech complexes were developed independently by Pavel Aleksandrov [6] and Eduard Čech [16][1]; despite Aleksandrov's earlier work, they are more commonly known as ***Čech complexes***.

The Aleksandrov-Čech complex captures almost all the topology of the union of ε-balls, thanks to the following seminal result:

The Nerve Lemma (Leray [40]) . *Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a finite set of open sets, such that the intersection of any subset of $\mathcal{U}$ is either empty or contractible. (In particular, each set U_i is contractible.) Then the nerve of $\mathcal{U}$ is homotopy equivalent to the union of sets in $\mathcal{U}$.*

[1] Both Aleksandrov and Čech actually considered a more general construction. Let $\mathcal{U} = \{U_1, U_2, \ldots\}$ be a set of open sets that cover some topological space X. The ***nerve*** of $\mathcal{U}$ is the abstract simplicial complex whose vertices are sets in $\mathcal{U}$, and whose simplices are finite subsets of $\mathcal{U}$ whose intersection contains a common point in X.

Corollary 15.1. *For any points set P and radius ε, the Aleksandrov-Čech complex $A\check{C}_\varepsilon(P)$ is homotopy-equivalent to the union of balls of radius ε centered at points in P.*



Aleksandrov-Čech complexes and unions of balls for two different radii. 2-simplices are yellow; 3-simplices are green.

15.1.2 Vietoris-Rips Complexes: Flags and Shadows

The ***proximity graph*** $N_\varepsilon(P)$ is the geometric graph whose vertices are the points P and whose edges join all pairs of points at distance at most 2ε; in other words, $N_\varepsilon(P)$ is the 1-skeleton of the Aleksandrov-Čech complex. The ***Vietoris-Rips complex*** $VR_\varepsilon(P)$ is the ***flag complex*** or ***clique complex*** of the proximity graph $N_\varepsilon(P)$. A set of $k+1$ points in P defines a k-simplex in $VR_\varepsilon(P)$ if and only if every pair defines an edge in $N_\varepsilon(P)$, or equivalently, if the set has diameter at most 2ε. Again, the Vietoris-Rips complex is an *abstract* simplicial complex.

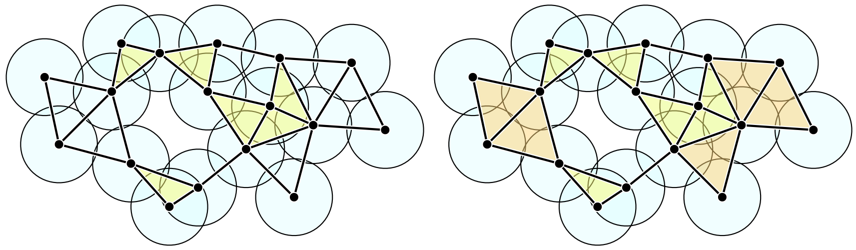
The Vietoris-Rips complex was used by Leopold Vietoris [57] in the early days of homology theory as a means of creating finite simplicial models of metric spaces.[2] The complex was rediscovered by Eliayu Rips in the 1980s and popularized by Mikhail Gromov [35] as a means of building simplicial models for group actions. 'Rips complexes' are now a standard tool in geometric and combinatorial group theory.

The triangle inequality immediately implies the nesting relationship $A\check{C}_\varepsilon(P) \subseteq VR_\varepsilon(P) \subseteq A\check{C}_{2\varepsilon}(P)$ for any ε, where $\subseteq$ indicates containment *as abstract simplicial complexes*. The upper radius 2ε can be reduced to $\sqrt{3}\varepsilon/2$ if the underlying metric space is Euclidean [21], but for arbitrary metric spaces, these bounds cannot be improved.

One big advantage of Vietoris-Rips complexes is that they determined entirely by their underlying proximity graphs; thus, they can be applied in contexts like sensor-network modeling where the underlying metric is unknown. In contrast, the Aleksandrov-Čech complex also depends on the metric of the ambient space that contains P; even if we assume that the underlying space is Euclidean, we need the lengths of the edges of the proximity complex to reconstruct the Aleksandrov-Čech complex.

On the other hand, there is no result like the Nerve Lemma for flag complexes. Indeed, it is easy to construct Vietoris-Rips complexes for points *in the Euclidean plane* that contain topological features of arbitrarily high dimension.

[2] Vietoris actually defined a slightly different complex. Let $\mathcal{U} = \{U_1, U_2, \ldots\}$ be a set of open sets that cover some topological space X. The ***Vietoris complex*** of $\mathcal{U}$ is the abstract simplicial complex whose vertices are points in X, and whose simplices are finite subsets of X that lie in some common set U_i. Thus, the Vietoris complex of an open cover is the dual of its Aleskandrov-Čech nerve. Dowker [25] proved that these two simplicial complexes have isomorphic homology groups.



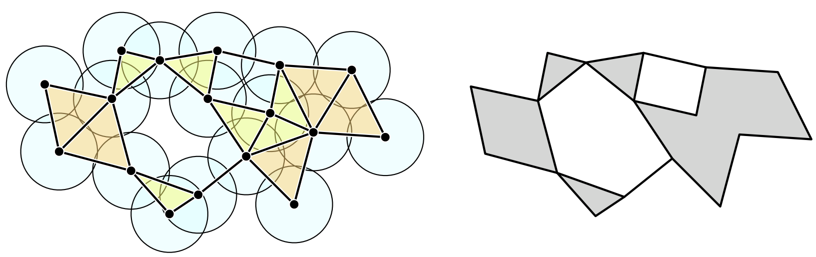
The Aleksandrov-Čech and Vietoris-Rips complexes for the same points and radius.

However, a few weaker results are known. A ***δ-sample*** of a metric space M is a point set P such that for any point $x \in M$, there is a point $p \in P$ whose distance from x (in the metric of M) is at most δ.

Theorem 15.2 (Latschev [39]). *Let M be a closed Riemannian manifold (for example, a smooth surface in $\mathbb{R}^d$). For any sufficiently small radius $\varepsilon > 0$, there is a sampling distance $\delta > 0$ such that for any δ-sample P of M, the Vietoris-Rips complex $VR_\varepsilon(P)$ is homotopy-equivalent to M.*

Suppose the points P are taken from some Euclidean space $\mathbb{R}^d$. For *any* simplicial complex Δ over the points P, there is a natural projection map from $|X|$ to $\mathbb{R}^d$, which maps each simplex in (a geometric realization of) X to the convex hull of its vertices (points in P). The image of this map is called the ***shadow*** of Δ. For points in the plane, the Vietoris-Rips shadow is a degenerate polygon with holes, possibly with hanging edges and isolated points.

Theorem 15.3 (Chambers *et al.* [18]). *For any point set P in the plane and any real number $\varepsilon > 0$, a cycle in $VR_\varepsilon(P)$ is contractible if and only if its projection is contractible in the shadow of $VR_\varepsilon(P)$.*

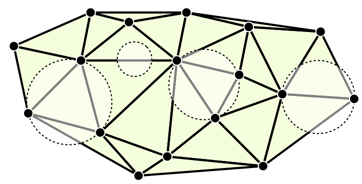


A Vietoris-Rips complex and its shadow.

15.1.3 Delaunay and Alpha Complexes: Empty Balls

A ***Delaunay ball*** for P is a closed ball that has no points of P in its interior. For any Delaunay ball B, The convex hull of $B \cap P = \partial B \cap P$ is called a ***Delaunay cell*** for P. Every Delaunay cell is a convex polytope, and it is not hard to show that the intersection of any two Delaunay cells intersect is a face of both. Thus, the set of Delaunay cells defines a polytopal complex, called the ***Delaunay complex***. The union of the cells in the Delaunay complex of P is the convex hull of P. If the points are in general position (at most $d+2$ points on any sphere), every Delaunay cell is a simplex; thus, for almost all point sets, the Delaunay complex is called the ***Delaunay triangulation***. Delaunay triangulations were first described by Boris Delone [24]; they are now arguably the most well-studied and widely applied structure in computational geometry.

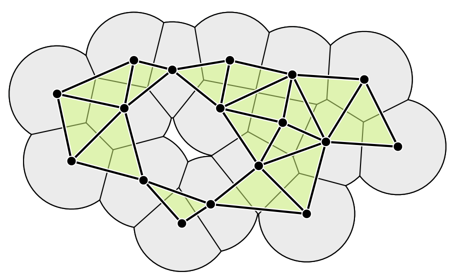
Fix a real radius $\varepsilon > 0$. For each point $p \in P$, let $\check{B}_\varepsilon(p)$ denote the set of points in the underlying space whose nearest neighbor in P is p and whose distance to p is at most $\varepsilon/2$. Thus, $\check{B}_\varepsilon(p)$ is the



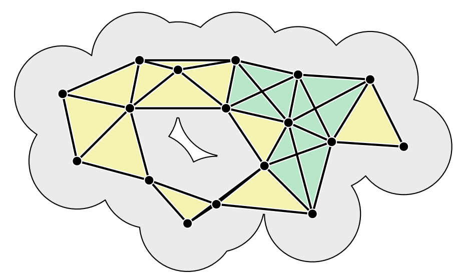
A Delaunay triangulation, with four Delaunay balls emphasized.

intersection of the ε-ball centered at p and the ***Voronoi region*** of p. The regions $\check{B}_\varepsilon(p)$ exactly cover the union of ε-balls centered at points in P. The ***alpha complex*** $\boldsymbol{\alpha_\varepsilon(P)}$ is the intersection complex of the set $\{\check{B}_\varepsilon(p) \mid p \in P\}$. The underlying space $|\alpha_\varepsilon(P)|$ is called an ***alpha shape*** of P.[3] The Nerve Lemma immediately implies that the alpha shape is homotopy equivalent to the union of the ε-balls; see also Edelsbrunner [26] for a self-contained proof.

If the point set P is in general position, the alpha complex $\alpha_\varepsilon(P)$ can also be defined as the intersection of the Delaunay triangulation of P and the Aleksandrov-Čech complex $A\check{C}_\varepsilon(P)$. Thus, $k+1$ points in P define a simplex in the alpha complex if and only if they lie in a closed ball B with diameter at most ε that contains no other point in P.



An alpha complex and a decomposed union of balls.



The corresponding Aleksandrov-Čech complex.

Alpha shapes were introduced by Edelsbrunner, Kirkpatrick, and Seidel, but only for points in the plane [27]; they were later generalized to points in $\mathbb{R}^3$ by Edelsbrunner and Mücke [28] and to weighted points in any Euclidean space by Edelsbrunner [26]. Of course, the definition is sensible for points in any metric space.

15.1.4 Witness Complexes

Witness complexes were introduced by Carlsson and de Silva [15, 19, 20] as 'weak' versions of the Delaunay complex. **⟨⟨Maybe next time; sorry, Vin!⟩⟩**

15.2 Configuration/State Complexes

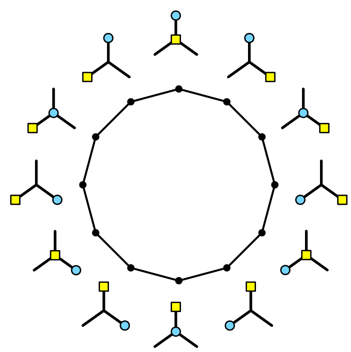
The following more abstract example was proposed by Abrams [3], modifying a similar construction by Ghrist and Kodischek [32, 31, 34]; see also Abrams and Ghrist [1, 2]. Imagine a set of k distinguished points, called ***agents***, located on the vertices and edges of a graph G, subject to the following rules designed to prevent collisions:

- If an agent is located at a vertex v, no other agent is located at v or inside any edge incident to v.
- If an agent is located inside an edge e, no other agent is located in e or at its endpoints.

[3] Originally, these were called the α-complex and α-shape, where α denoted the proximity radius. Unfortunately, this usage leads to considerable confusion if α is set to any particular value—What's a $\sqrt{2}$-complex?

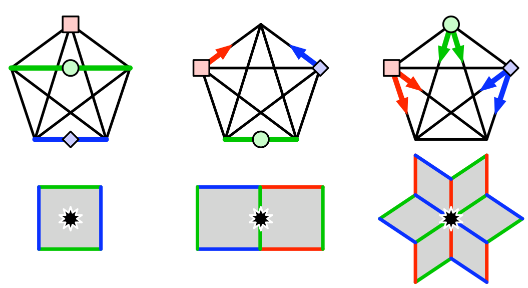
The set of all legal configurations of k agents on the graph G has a natural description as a *cube complex* $\boldsymbol{D^k(G)}$, called a ***(discrete) configuration complex***. Specifically, let G^k be the cube complex whose cells are all possible k-fold Cartesian products of vertices and edges in G. Then $D^k(G)$ is the subcomplex of G^k in which the k factors of each cell are distinct. That is, each cell in $D^k(G)$ is the product $x_1 \times x_2 \times \cdots \times x_k$, where each component x_i is either a vertex or an edge of G, and there is a legal configuration with the ith agent located on x_i. The dimension of the cube is the number of factors that are edges.

A few examples will hopefully make this definition clear. First consider the complex $D^2(Y)$, where Y is a tree with one degree-3 vertex and three leaves. There are 24 legal configurations of two agents on Y: 12 with both agents on vertices, and 12 with one agent on a vertex and the other on an edge. These respectively determine the vertices and edges of $D^2(Y)$. Each vertex-vertex configuration is adjacent to exactly two vertex-edge configurations, and it is possible to move continuously the agents from any legal configuration to any other through only legal configurations. Thus, $D^2(Y)$ consists of a single cycle of length 12.



$D^2(Y)$

As a more interesting example, consider $D^3(K_5)$, where K_5 is as usual the complete graph with 5 vertices. This complex has 60 0-cells (all three agents on vertices), 180 1-cells (two on vertices, one inside an edge), and 90 2-cells (one on a vertex, two inside edges); thus, $\chi(D^3(K_5)) = -30$. We easily observe that each 1-cell is incident to exactly two 2-cells, and each 0-cell is incident to six 1-cells. It follows immediately that $D^3(K_5)$ is a 2-manifold![4] It is easy to reach any configuration from any other, so $D^3(K_5)$ is connected; with more work, one can prove that $D^3(K_5)$ is orientable. Thus, $D^3(K_5)$ is a quadrangulation of the orientable 2-manifold of genus 16.



A 2-cell in $C^3(K_5)$, two 2-cells incident to a 1-cell, and six 1-cells incident to a 0-cell.

The complexes $D^k(G)$ are particularly nice examples of a general class of ***discrete configuration complexes*** or ***state complexes***, developed in detail by Ghrist and Peterson [33]. A completely rigorous definition is quite complicated, but intuitively, a ***configuration*** is a (typically finite) set of objects from some (typically finite) discrete space, where the objects satisfy certain combinatorial constraints. A

[4]These examples are chosen very carefully; for almost all graphs G and integers k, the complex $D^k(G)$ is *not* a manifold.

transition modifies the configuration by adding, deleting, or replacing a constant number of objects in the configuration. A set of transitions is ***independent*** if the transitions can be applied in any order, always achieving the same result. Finally, a discrete configuration complex is a cube complex whose vertices are the configurations, whose edges are transitions, and whose k-dimensional cells are defined by independent sets of k transitions.

15.3 The Monotone Property Complex

An ***n-vertex graph property*** is a subset of the set of all n-vertex graphs. Equivalently a graph property is a function of the form $\Pi\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$, where the input bits represent the entries in the adjacency matrix of an undirected graph, that remains unchanged by any permutation of the vertices. Thus, we are concerned with properties of *unlabeled* graphs (connectivity, planarity, hamiltonicity) as opposed to properties of labeled graphs (vertex 17 has degree 42). A graph property is ***nontrivial*** if at least one graph has the property and at least one graph does not. A graph property is ***downward monotone*** if every subgraph of a graph with the property also has the property, and ***upward monotone*** if every supergraph of a graph with the property also has the property.

A graph property is ***evasive*** if every algorithm to determine whether a graph has the property, given the graph's adjacency matrix, must examine every bit in the input in the worst case. More succinctly, a graph property is evasive if its deterministic decision tree complexity is exactly $\binom{n}{2}$. Several authors developed proofs of evasiveness for specific graph properties, such as connectedness, acyclicity, planarity, and containing a fixed complete subgraph [9, 7, 37, 46].

Aanderaa-Karp-Rosenberg Conjecture [54]. *Every nontrivial monotone graph property is evasive.*

Motivated by results of Holt and Reingold [37] on the evasiveness of strong connectivity and acyclicity, Aanderaa and Rosenberg originally conjectured that testing any nontrivial monotone graph property requires examining $\Omega(n^2)$ bits. This weaker conjecture was almost immediately proved by Rivest and Vuillemin [53]. The stronger conjecture, which Rosenberg attributes to Karp, is still open.

A significant step toward settling the stronger conjecture was taken by Kahn, Saks, and Sturtevant [38], who proved that any nontrivial monotone property of n-vertex graphs is evasive when n is a prime power. Their key insight is that every nontrivial downward-monotone graph property Π for n-vertex graphs can be viewed as an abstract simplicial complex $\Delta(\Pi)$ with $\binom{n}{2}$ vertices. Their key lemma is the following:

Lemma 15.4. *If Π is a non-evasive property of n-vertex graphs, then $\Delta(\Pi)$ is collapsible (and thus contractible).*

Yao [58] generalized the results of Kahn, Saks, and Sturtevant to prove that all nontrivial monotone properties of *bipartite* graphs are evasive. Chakrabati, Khot, and Shi [17] generalized the approach even further, showing that the property of containing a fixed subgraph is evasive for an arithmetic sequence of values of n, and that membership in a minor-closed family is evaisve for all sufficiently large n.

15.4 Presentation Complexes and Undecidability

Let $G = \langle \Sigma \mid R \rangle$ be a finitely presented group, with generators $\Sigma = \{x_1, \ldots, x_n\}$ and relators $R = \{r_1, \ldots, r_k\}$, where each r_i is a word in $(\Sigma^{\pm})^*$. The ***presentation complex $K(G)$*** is a two-dimensional CW-complex with one vertex $\bullet$, one edge for each generator x_i, and one 2-cell for each relator r_i, where the word r_i describes the gluing map for the corresponding disk. For example, every one-vertex polygonal schema is a presentation complex. It is easy to prove that the fundamental group $\pi_1(K(G), \bullet)$

is isomorphic to the given group G. Markov [44, 43, 45] used presentation complexes to prove that a huge number of topological problems are formally undecidable for arbitrary cell complexes, including many problems for which we have already seen algorithms when the input is a 2-manifold.

All of Markov's undecidability results trace back to Turing's proof that the halting problem for Turing machines is undecidable [55] (which in turn is an intellectual descendant of Cantor's diagonal argument that the real numbers are uncountable [14]). In 1950, Turing proved that the word problem is undecidable for certain finitely presented semigroups (groups without inverses) [56], by reduction from the halting problem. Building on Turing's results, Novikov [47, 48, 49] proved the undecidability of the word and conjugacy problems for finitely presented groups, originally proposed by Dehn [22, 23]; these result were also independently discovered by Boone [10, 11, 12] and Britton [13]. In particular, there is a *specific* finitely-presented group $G_0 = \langle \Sigma \mid R \rangle$ such that a word $w \in (\Sigma^{\pm})^*$ represents the trivial element of G_0 if and only if is an encoding of a Turing machine that accepts the empty string. Thus, it is undecidable whether a given cycle in the presentation complex $K(G)$ is contractible.

Adyan [4, 5] (in his PhD thesis under Novikov) and Rabin [51, 52] independently proved that all nontrivial algebraic properties[5] of finitely presented groups are undecidable, generalizing earlier results for finitely presented semigroups by Markov [41, 42].[6] In particular, there is no algorithm to decide whether a given finitely-presented group is trivial. It immediately follows that there is no algorithm to decide whether a given presentation complex $K(G)$ has a trivial fundamental group.

The second barycentric subdivision $Sd^2K(G) = Sd(Sd(K(G)))$ is a simplicial complex; thus, all these undecidability results extend to simplicial complexes. Menger's embedding theorem implies that $Sd^2K(G)$ can be linearly embedded in $\mathbb{R}^5$; let $\bar{K}_\varepsilon(G) \subset \mathbb{R}^5$ be the set of points within distance ε of the image of such an embedding. For all sufficiently small $\varepsilon > 0$, the space $\bar{K}_\varepsilon(G)$ is a 5-manifold with boundary that is homotopy equivalent to $K(G)$; in particular, $\pi_1(\bar{K}_\varepsilon(G)) \cong G$. Markov [44, 43, 45] gave a more careful construction of a 4-manifold whose fundamental group is any given finitely-presented group. Thus, most questions about homotopy in/between triangulated *manifolds* are undecidable in dimensions 4 and higher.

Theorem 15.5. *The following problems are undecidable:*

- *Given a cycle in a manifold simplicial complex, is it contractible?*
- *Given two cycles in a manifold simplicial complex, are they homotopic?*
- *Given a simplicial complex, is it contractible?*
- *Given a manifold simplicial complex, is it homeomorphic (or even homotopy-equivalent) to a particular 4-manifold M_4?*
- *Given a manifold simplicial complex, is it homeomorphic (or even homotopy-equivalent) to **any** fixed 5-manifold (for example, S^5)?*
- *Given a simplicial complex, is it a 6-manifold?*

15.5 More cool examples I didn't have time to talk about

- Pocchiola-Vegter visibility complex [50]
- Billera-Holmes-Vogtmann phylogenetic tree complex [8]

[5] A property of groups is ***algebraic*** if it is invariant under group isomorphisms; thus, we are concerned with properties of the group, not properties of its presentation. An algebraic property is ***nontrivial*** if some finitely presented group has the property, and there is a finitely presented group that is not a subgroup of a group that has the property. Nontrivial algebraic properties are sometimes called ***Markov properties***.

[6] Markov's semigroup results are nearly equivalent to ***Rice's theorem***, which states that any nontrivial property of partial functions (and thus any nontrivial property of Turing machines) is undecidable.

- Herlihy-Shavit protocol complex [36]
- Erdmann strategy complex [29, 30]

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