

One who does not realize his own value is condemned to utter failure.
(Every kind of complex, superiority or inferiority, is harmful to man).

— ‘Alī ibn Abī Ṭālib, *Nahj al-Balagha* [Peak of Eloquence]

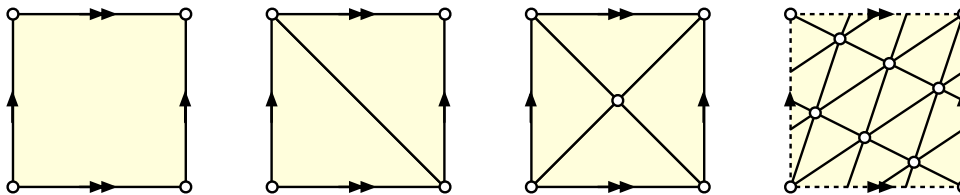
There once lived a man named Oedipus Rex.
You may have heard about his odd complex.
His name appears in Freud’s index
‘Cause he loved his mother.

— Tom Lehrer, “Oedipus Rex”, *More of Tom Lehrer* (1959)

15 Cell Complexes: Definitions

With a few exceptions, we have focused exclusively on problems involving 2-manifolds, represented by polygonal schemata. Before we (finally!) consider problems involving more general spaces, we need to step back and consider how these more general spaces might be represented, as part of the input to an algorithm. This is a surprisingly subtle question, even if we restrict our attention to higher-dimensional manifolds.

A large class of topological spaces of practical interest can be represented by a decomposition into subsets, each with simple topology, glued together ‘nicely’ along their boundaries. A decomposition of this form is commonly called a **cell complex**. Abstract graphs (as branched 1-manifolds) are examples of cell complexes, as are polygonal schemata, quadrilateral and hexahedral meshes, and 3d triangulations supporting normal surfaces. There are several different ways to formalize the intuitive notion of a ‘cell complex’, striking different balances between simplicity and generality. I’ll describe several different definitions in this lecture. The figure below shows the ‘cell complex’ of minimum complexity (number of cells) homeomorphic to the torus, for four standard definitions.



The simplest CW complex, Δ -complex, regular Δ -complex, and simplicial complex homeomorphic to the torus.

15.1 Abstract Simplicial Complexes

An **abstract simplicial complex** is a collection \mathcal{X} of finite sets that is closed under taking subsets; that is, for any set $S \in \mathcal{X}$ and any subset $T \subseteq S$, we have $T \in \mathcal{X}$. In particular, the intersection of any two sets in \mathcal{X} is also a set in \mathcal{X} , and the empty set \emptyset is an element of every simplicial complex.

The sets in \mathcal{X} are called **simplices**. The **vertices** of \mathcal{X} are the singleton sets in \mathcal{X} (or their elements); the set of vertices of \mathcal{X} is denoted \mathcal{X}_0 . The **dimension** of a simplex is one less than its cardinality; in particular, every vertex is a 0-dimensional simplex, and the empty set has dimension -1 . The **dimension** of a simplicial complex is the largest dimension of any simplex. The subsets of a simplex are called its **faces**. A simplex in \mathcal{X} is **maximal** if it is not a proper face of any other simplex in \mathcal{X} ; the maximal simplices in \mathcal{X} are also called the **facets** of \mathcal{X} . An abstract simplicial complex \mathcal{X} is **pure** if every facet has the same dimension.

For example, the power set of any finite set is a pure abstract simplicial complex. An undirected (simple) graph with no isolated vertices is a pure 2-dimensional abstract simplicial complex. *Matroids*

are finite pure abstract simplicial complexes with an additional exchange property.¹ The vertex-set of an abstract simplicial complex need not be finite, or even countable; for example, consider the set of all finite subsets of points in the plane, in which every pair of points has distance at most 1. (This is an example of a **Vietoris-Rips complex**, which we will see in more detail in the next lecture.)

Abstract simplicial complexes are also known to combinatorists as *independence systems*, *hereditary set systems*, and *downward-closed set systems*; they were first proposed as models of topological spaces by Pavel Alexandrov [1].²

15.2 Geometric Simplicial Complexes

Abstract simplicial complexes are not interesting topological spaces by themselves; they're just sets of finite sets! However, they can be used to describe the combinatorial structure of many topological spaces. Any abstract simplicial complex \mathcal{X} has a **geometric realization**, defined by mapping the vertices of \mathcal{X} to generic points in a sufficiently high-dimensional space. Geometric simplicial complexes were first formalized in the seminal work of Poincaré [5].

A **hyperplane** in \mathbb{R}^D is the set of points $\{x \mid \langle a, x \rangle = b\}$ for some vector $a \in \mathbb{R}^D$ and some scalar $b \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ denotes inner product. A set of points in \mathbb{R}^D is **affinely independent** if it is not contained in a hyperplane; an affinely independent set in \mathbb{R}^d contains at most $D + 1$ points. The **convex hull** of a finite set $X \subset \mathbb{R}^D$ is the set of all weighted averages of points in X :

$$\mathbf{conv}(X) := \left\{ \sum_{x \in X} \lambda_x x \mid \sum_{x \in X} \lambda_x = 1 \text{ and } 0 \leq \lambda_x \leq 1 \text{ for all } x \in X \right\}$$

The convex hull of a finite set of points is called a **polytope**. A hyperplane **supports** a polytope if it does not intersect its interior. The intersection of a polytope and any supporting hyperplane is a **proper face** of the polytope. It is easy to check that $\mathbf{conv}(X) \cap h = \mathbf{conv}(X \cap h)$ for any point set X and supporting hyperplane h ; thus, proper faces of polytopes are also polytopes. In particular, the empty set is a polytope!

A **k -simplex** is the convex hull of a set of $k + 1$ affinely independent points, called its **vertices**; a **simplex** is a k -simplex for some integer k . For example, a tetrahedron is a 3-simplex, a triangle is a 2-simplex, a line segment is a 1-simplex, a point is a 0-simplex, and the empty set is the unique (-1) -simplex. A **face** of a simplex is the convex hull of a subset of its vertices; in particular, a **facet** of a k -simplex is the convex hull of all but one of its vertices. Thus, every k -simplex has exactly $k + 1$ facets and exactly 2^k faces. Every face of a simplex is a simplex.

A **geometric simplicial complex** is a set Δ of simplices in some Euclidean space \mathbb{R}^D , satisfying two conditions: (1) every face of a simplex in Δ is also in Δ , and (2) the intersection of any two simplices in Δ is a face of both.³ The simplices in Δ are called its **cells**. For example, the set of faces of any simplex define a simplicial complex.

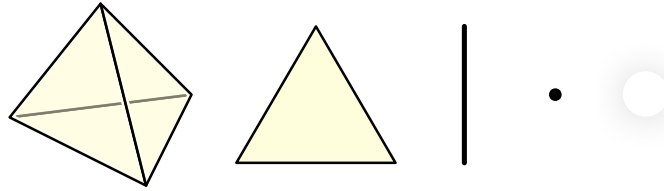
¹A pure abstract simplicial complex \mathcal{M} is a matroid if, for any two simplices $A, B \in \mathcal{M}$ such that $|A| > |B|$, there is an element $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{M}$.

²In Alexandrov's own words [1, p. 490]:

Es soll von Anfang an folgende Bemerkung gemacht werden. In der Topologie, ebenso wie in der elementaren Geometrie, ist ein n -dimensionaler Simplex immer durch seine $n + 1$ Eckpunkte vollständig bestimmt, so daß zwei dieselben Eckpunkte besitzende Simplexe untereinander identisch sind.

Daraus folgt, daß der *topologische Simplex nichts anderes als das System seiner $n + 1$ Eckpunkte ist*, also eigentlich einer endliche Menge von Elementen (eines bestimmten Elementenvorrates) die „Eckpunkte“ heißen. Die Dimension des Simplexes ist einfach die um eine Einheit verminderte Kardinalzahl dieser Menge; die Seiten des Simplexes sind Teilmengen derselben endlichen Menge.

³Equivalently: (2') the simplices in Δ have disjoint interiors.



A 3-simplex, a 2-simplex, a 1-simplex, a 0-simplex, and the (-1)-simplex

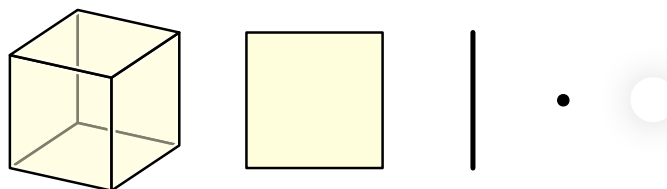
The **underlying space** (or *carrier* or *polyhedron*) of a simplicial complex Δ , denoted $|\Delta|$, is the union of its simplices. The simplicial complex Δ is considered a **triangulation** of any space homeomorphic to $|\Delta|$. It is fairly common practice to blur (or completely ignore) the distinction between a simplicial complex and its underlying space.

Finally, a **geometric realization** $\Delta(\mathcal{X}, f)$ of an abstract simplicial complex \mathcal{X} is defined by a function $f: \mathcal{X}_0 \rightarrow \mathbb{R}^D$ that maps each simplex in \mathcal{X} to an affinely independent set in \mathbb{R}^D , such that . It is easy to see that any *finite* abstract simplicial complex has a geometric realization into a space of dimension $|\mathcal{X}_0|$. More complicated arguments, ultimately to to Menger, imply that any d -dimensional abstract simplicial complex can be geometrically realized in \mathbb{R}^{2d+1} ; for example every graph can be embedded in \mathbb{R}^3 with disjoint line segments as edges. In fact, every finite abstract simplicial complex has *infinitely many* geometric realizations, but they are all homeomorphic to each other. Thus, the **underlying space** $|\mathcal{X}|$ of an *abstract* simplicial complex \mathcal{X} can be defined as the underlying space of *any* geometric realization of \mathcal{X} . Conversely, every geometric simplicial complex Δ is a geometric realization of a unique abstract simplicial complex $\mathcal{X}(\Delta)$. Indeed, this correspondence is the basis of most file formats for triangulated surfaces and tetrahedral finite-element meshes.

15.3 Polytopal Complexes

Polytopal complexes (also called **polyhedral complexes**) are a natural generalization of geometric simplicial complexes, first proposed by Poincaré [4] even before his use of simplicial complexes. A polyhedral complex is a set Π of polytopes in some Euclidean space \mathbb{R}^D , satisfying two conditions: (1) every face of a polytope in Π is also in Π , and (2) the intersection of any two polytopes in Π is a face of both. The polytopes in Π are called its **cells**. For example, the set of all faces of any convex polytope define a polytopal complex. A geometric simplicial complex is simply a polytopal complex in which every cell is a simplex.

Another natural special case of polytopal complexes uses cubes everywhere instead of simplices. A **k -cube** is a polytope that is combinatorially isomorphic to the standard cube $\square_k = [0, 1]^k$. Every k -cube has exactly $2k$ facets, 2^k vertices, and $3^k + 1$ faces overall; every face of a k -cube is a j -cube for some integer $j \leq k$. A **geometric cube complex** is a polytopal complex in which every k -dimensional cell is a k -cube. Quadrilateral and hexahedral meshes are examples of geometric cube complexes.



A 3-cube, a 2-cube, a 1-cube, a 0-cube, and the (-1)-cube

15.4 Δ -Complexes

The definition of geometric simplicial complexes has two unfortunate features. First, the definition requires an ambient space \mathbb{R}^D ; thus, formally, a triangulation of an abstract 2-manifold is *not* a geometric simplicial complex. Second, the definition requires each k simplex to have $k+1$ *distinct* vertices, and each set of $k+1$ vertices to define *at most one* k -simplex. This condition seems unnecessarily restrictive, since it forbids structures like (triangulated) systems of loops that have already proved useful. **Δ -complexes** bypass both of these restrictions; intuitively, a Δ -complex is a topological spaces constructed by gluing simplices together along their boundaries.

Δ -complexes originate with the work of Eilenberg and Zilber [2], who proposed an equivalent structure called a (*complete*) *semi-simplicial complex*, later abbreviated to *css complex* by other authors; css complexes are now more commonly known as (*semi*-)*simplicial sets*. The name ‘ Δ -complex’ was proposed by Hatcher [3].

For any non-negative integer n , the **standard n -simplex** Δ_n is the convex hull of the positive unit coordinate vectors in \mathbb{R}^{n+1} :

$$\Delta_n := \left\{ (\lambda_0, \lambda_1, \dots, \lambda_n) \mid \sum_{i=0}^n \lambda_i = 1 \text{ and } 0 \leq \lambda_i \leq 1 \text{ for all } i \right\}$$

An **n -simplex** is the image of Δ_n under a homeomorphism; in particular, every k -dimensional face of Δ_n is a k -simplex. For any integer $-1 \leq k \leq n$, let $\Delta_n^{(k)}$ denote the union of all $\binom{n}{k+1}$ k -dimensional faces of Δ_n . In particular, $\Delta_n^{(-1)} = \emptyset$, $\Delta_n^{(0)}$ is a discrete set of $n+1$ points, and $\Delta_n^{(n)} = \Delta_n$. We also call $\Delta_n^{(n-1)}$ the **boundary** of Δ_n , denoted $\partial\Delta_n$.

A Δ -complex X is the last space⁴ in an inductively defined sequence of topological spaces $X^{(0)} \subseteq X^{(1)} \subseteq \dots \subseteq X^{(n)} = X$; each space $X^{(k)}$ is called the **k -skeleton** of X . The 0-skeleton $X^{(0)}$ is just a discrete set of points, called the **vertices** of X . For each integer $k > 0$, we inductively construct the **k -skeleton** $X^{(k)}$ by attaching a set of k -simplices to the $(k-1)$ -skeleton $X^{(k-1)}$. Each k -simplex Δ_k is attached by a **gluing map** $\sigma: \partial\Delta_k \rightarrow X^{(k-1)}$ that maps the interior of each face of Δ_k homeomorphically to the interior of a simplex in $X^{(k-1)}$ of the same dimension. Thus, σ maps each vertex of Δ_k to a point in $X^{(0)}$, each edge of Δ_k to an edge in $X^{(1)}$, and so on; more briefly, we say that each gluing map σ is **simplicial** or **cellular**. More formally, $X^{(k)}$ is defined as the quotient space

$$X^{(k)} := (X^{(k-1)} \sqcup \bigsqcup_{i \in I_k} \Delta_{k,i}) / \sim,$$

for some index set I_k , where \sqcup denotes disjoint union, each $\Delta_{k,i}$ is a disjoint copy of Δ_k , and $x \sim \sigma_i(x)$ for each gluing map $\sigma_i: \partial\Delta_{k,i} \rightarrow X^{(k-1)}$. The interiors of the simplices $\Delta_{k,i}$ are the **k -cells** of X .

We call a Δ -complex X **regular** if every k -cell in X has $k+1$ distinct vertices, for all k . A regular Δ -complex is **proper** (or **simplicial**) if each subset of $k+1$ vertices is incident to at most one k -cell, for all k , or equivalently, if the intersection of any two cells in X is (the image of) a face of both cells. A tedious inductive argument implies that every proper Δ -complex is homeomorphic to (the underlying space of) a geometric simplicial complex, by a homeomorphism that maps simplices to simplices, and vice versa. Thus, we usually use the term **simplicial complex** to refer either to a geometric simplicial complex or to a proper Δ -complex.

However, not every Δ -complex is (equivalent) to a simplicial complex. Consider the triangulations of the torus shown on the

⁴Infinite-dimensional Δ -complexes can be defined by setting $X = \bigcup_{k \geq 0} X^{(k)}$ with the weak topology: A set $A \subseteq X$ is open if and only if $A \cap X^{(k)}$ is open for all k .

The **barycentric subdivision** $Sd(\Delta_n)$ of the standard simplex Δ_n is a simplicial complex defined inductively by joining the centroid of Δ_n to the barycentric subdivision of the facets of $\partial\Delta_n$. (As a base case, take $Sd(\Delta_0) = \Delta_0$, or even $Sd(\Delta_{-1}) = \emptyset$.) The complex $Sd(\Delta_n)$ has $2^{n+1} - 1$ vertices, located at the centroids of the non-empty faces of Δ_n , and $(n + 1)!$ facets, one for each maximal inclusion-chain of faces of Δ_n . For any Δ -complex X , the barycentric subdivision $Sd(X)$ is defined by replacing each simplex in X with its barycentric subdivision. Although X and $Sd(X)$ are homeomorphic as topological spaces, the nicer combinatorial structure of $Sd(X)$, as described in the following lemma, allows us to simplify several proofs.

Lemma 15.1. *The barycentric subdivision of any Δ -complex is a regular Δ -complex. The barycentric subdivision of any regular Δ -complex is a proper Δ -complex.*

15.5 CW-complexes

We can of course generalize Δ -complexes by allowing more general shapes for each cell. For example, by replacing simplices with cubes, we obtain **cube complexes**; by replacing simplices with arbitrary polytopes, we obtain **polytopal complexes**. These more general structures still require **cellular** gluing maps, which map the interior of each k -dimensional face of any cell homeomorphically to the interior of a k -dimensional cell.

But the most general notion of ‘cell complex’ arises by simply removing the requirement that the gluing maps are cellular. The resulting structure was first proposed by Whitehead, originally as *membrane complexes* [6, 7], but later under the still-standard name **closure finite⁵ complexes with weak topology⁶**, or more simply **CW complexes** [8]. (Indeed, some authors refer to Δ -complexes as simplicial CW complexes.)

Because the facial structure of the cells is irrelevant for CW complexes, they are typically defined in terms of the standard unit balls $B^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. The boundary of the unit ball B^n is the standard unit sphere $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.

A (finite-dimensional) CW-complex X is the last of a nested sequence of topological spaces $X^{(0)} \subseteq X^{(1)} \subseteq \dots \subseteq X^{(n)} = X$ defined inductively as follows. The 0-skeleton $X^{(0)}$ is just a discrete set of points. For each integer $k > 0$, the k -skeleton $X^{(k)}$ is the result of attaching a set of k -dimensional balls to $X^{(k-1)}$ by gluing maps of the form $\sigma : S^{k-1} \rightarrow X^{(k-1)}$. For each integer k , the subspace $X^{(k)}$ is called the **k -skeleton** of X , and the interiors of the balls B^k attached to $X^{(k-1)}$ are the **k -cells** of X .

A CW complex is **regular** if its gluing maps are embeddings (homeomorphisms onto their images); in particular, each cell in a regular CW complex X meets each vertex of X at most once. A CW complex is **proper** if the intersection of any two closed cells is another closed cell.

15.6 Familiar Examples

We have already seen several simple examples of simplicial complexes, Δ -complexes, and CW-complexes. Simple graphs are 1-dimensional (abstract) simplicial complexes; graphs with parallel edges are regular Δ -complexes; graphs with parallel edges and loops are Δ -complexes. Polygonal schemata are 2-dimensional CW complexes. Surface triangulations are sometimes defined as simplicial complexes (via simple graphs), but most results about triangulations apply equally well to surface Δ -complexes. Quadrilateral and hexahedral meshes are proper (or geometric) cube complexes. The three-dimensional

⁵The closure of each cell intersects only finitely many other cells. This property is trivial for CW complexes with a finite number of cells; for infinite complexes, it follows by a compactness argument.

⁶A subset of X is closed if and only if its intersection with the closure of each cell in X is closed. This property is trivial for finite-dimensional CW complexes; for infinite-dimensional CW complexes, it is an essential part of the definition.

triangulations used in normal surface theory are usually described as simplicial complexes, but there is no danger in using Δ -complexes instead. We will see several more examples in the next lecture.

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