Q: So you like the grass?
A: Yeah. Well, actually I don't care what surface I'm playing on.
— Daniela Hantuchová (May 17, 2001)
press conference at the Tennis Masters Series, Rome, Italy

6 Surface Classification

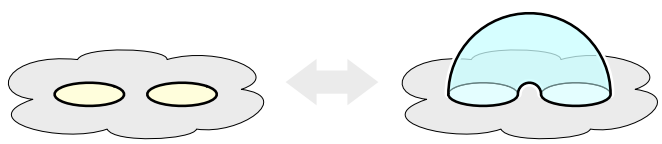
In this lecture, I'll give a proof of the most fundamental result in 2-manifold topology:

Surface Classification Theorem. *Every compact connected 2-manifold can be constructed from the sphere by attaching either a finite number or handles or a finite number of Möbius bands.*

A complete proof of this theorem relies on Kerékjártó and Rado's proof that any compact 2-manifold has a triangulation, but the classification of *triangulated* 2-manifolds is much older. Different sources attribute the first proof to Brahana [1], Dehn and Heegard [2], and Dyck [3].[1] Brahana's proof is the one that appears in most topology textbooks, thanks to its appearance in an early textbook of Siefert and Threlfall. Several other proofs are known; we refer in particular to a completely self-contained proof by Thomassen [10] and Conway's 'zero irrelevancy' proof [5].

6.1 Attaching Handles and Möbius Bands

To ***attach a handle*** to a surface Σ, find two disjoint closed disks on Σ, delete the interiors of the disks, and glue an annulus to the two boundary circles. The inverse operation—deleting an annulus and gluing disks onto each boundary circle—is called ***detaching a handle***.



Left to right: Attaching a handle. Right to left: Detaching a handle.

To ***attach a Möbius band*** to a surface Σ, find a single closed disk in Σ, delete its interior, and glue a Möbius band to its boundary circle. The inverse operation—deleting a Möbius band and gluing a disk onto its boundary circle—is called ***detaching a Möbius band***.

Left to right: Attaching a Möbius band. Right to left: Detaching a Möbius band.

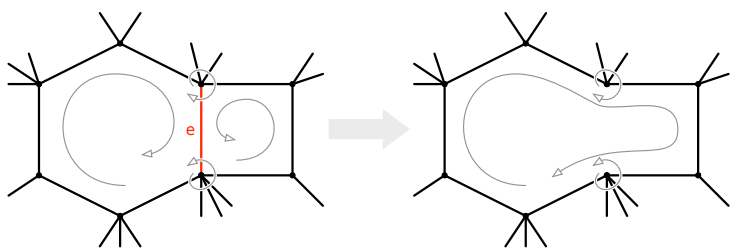
Let $\Sigma(g,h)$ denote the surface obtained from the sphere by attaching g handles and h Möbius bands. (You should convince yourself that it doesn't matter where the handles and Möbius bands are attached, or in which order.) For example, $\Sigma(0,0)$ is the sphere; $\Sigma(1,0)$ is the torus; $\Sigma(0,1)$ is the ***projective plane***; and $\Sigma(0,2)$ is the ***Klein bottle***.

[1] However, the classification of orientable surfaces was previously *stated* in various forms by Riemann, Klein, Jordan [6], and Möbius [9].

6.2 Edge Surgery

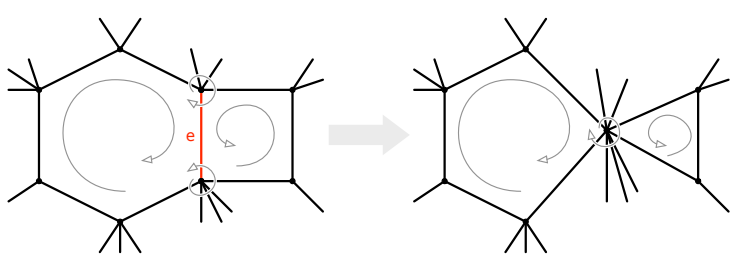
My proof of the classification theorem starts with a polygonal schema for the 2-manifold and modifies its graph through a series of ***edge contractions*** and ***edge deletions***. Both operations remove edges from the graph, but in different ways.

Let e be an edge in the graph G. If e separates two distinct faces f and f', we can ***delete*** e to obtain a new graph $\boldsymbol{G \setminus e}$ (pronounced 'G without e'). The two faces f and f' are merged into a single face in $G \setminus e$. If the embedding of G is represented by a rotation system, we simply delete both occurrences of e. If the embedding is represented by a polygonal schema, we merge the cyclic orders of edges around f and f'. If the cyclic orders traverse e in two different directions, we simply break the cycles at e and concatenate them. If both cyclic orders traverse e in the same direction, then we must ***flip*** one of the two faces before we merge. With an appropriate data structure for either representation, we can transform G into $G \setminus e$ in $O(1)$ time.



Deleting an edge separating two faces.

On the other hand, if e has two distinct endpoints u and v, we can ***contract*** e to obtain a new graph $\boldsymbol{G \,/\, e}$ (pronounced 'G mod e'). The two vertices u and v are merged into a single vertex in $G \,/\, e$. If the graph embedding is represented by a polygonal schema, we simply delete both occurrences of e. If the embedding is represented by a rotation, we merge the cyclic orders of edges around v and v'. If the cyclic orders cross e in two different directions, we simply break the cycles at e and concatenate them. If both cyclic orders cross e in the same direction, then we must ***flip*** one of the two vertices before we merge. With an appropriate data structure for either representation, we can transform G into $G \,/\, e$ in $O(1)$ time.



Contracting an edge joining two vertices.

The inverse of edge deletion is ***edge insertion***, and the inverse of edge contraction is ***edge expansion***. However, to fully specify an edge insertion, we must specify not only the endpoints of the new edge, but the face f being split by the new edge and the corners of f that the new edge will connect. Similarly, to fully specify an edge expansion, we must specify not only the faces on either side of the new edge, but the vertex v being split by the new edge and the partition of edges incident to v.

It should come as no surprise that edge contraction and edge deletion are dual to each other. In particular, we have identities $(G \setminus e)^* = G^* \,/\, e^*$ and $(G \,/\, e)^* = G^* \setminus e^*$.

For any disjoint subsets A and B of the edge of G, we let $G \,/\, A \setminus B$ denote the graph resulting from contracting every edge in A and contracting every edge in B. The order of contractions and deletions is unimportant; any permutation of the operations leads to the same graph.

A ***system of loops*** in a surface Σ is a cellularly embedded graph in Σ with exactly one vertex and exactly one face. The dual of a system of loops is a ***reduced polygonal schema***, which is (not surprisingly) a polygonal schema with one face, whose corners all identify to a single vertex.

Lemma 6.1. *Every compact connected 2-manifold has a system of loops.*

Proof: Let Σ be a compact connected 2-manifold. Let G be a graph cellularly embedded in Σ; such a graph exists by the Kerékjártó-Rado theorem. Fix an arbitrary spanning tree T of G. The graph $G \,/\, T$ obtained by contracting every edge in T is cellularly embedded in Σ and has exactly one vertex. Next, fix an arbitrary spanning tree C^* of the dual graph $(G \,/\, T)^* = G^* \setminus T^*$, and let C be the corresponding subgraph of $G \,/\, T$. The graph $G \,/\, T \setminus C$ is a system of loops. □

We obtain the same system of loops by contracting the edges in the spanning tree T and/or deleting the edges in the spanning ***cotree*** C in *any* order. Let L be the subset of *leftover* edges in G that survive in $G \,/\, T \setminus U$. The partition (T, L, C) of the edges of G is called a ***tree-cotree decomposition***.

6.3 The Proof

The proof of the surface classification theorem begins with a cruder classification.

Theorem 6.2. *Every compact connected 2-manifold is homeomorphic to $\Sigma(g,h)$, for some non-negative integers g and h.*

Proof: Let Σ be a compact connected 2-manifold. Let L be a system of loops in Σ with basepoint v, and let f denote the single face of L. The proof proceeds by induction on the number of edges in L. There are three cases to consider: either L is empty, L contains a 1-sided loop, or L is non-empty and contains only two-sided loops.

First, suppose L is the *empty* system of loops, consisting entirely of a single vertex v in Σ. Because L has a cellular embedding, the subspace $\Sigma \setminus \{v\}$ must be homeomorphic to the plane. It follows that Σ is homeomorphic to the sphere $\Sigma(0,0)$.[2] Conversely, the Jordan-Schönflies theorem implies that every simple loop separates the sphere into two disks; thus, any system of loops on the sphere must be empty.

Now suppose L contains a one-sided loop ℓ. We modify the graph L and the underlying surface Σ in four stages.

- First, we expand the vertex v into two vertices $v^\flat$ and $v^\sharp$ joined by an edge e. The loop ℓ becomes an edge from $v^\flat$ to $v^\sharp$. The two ends of ℓ partition the cyclic sequence of edges incident to v into two intervals. One interval becomes the order around $v^\flat$; the other becomes the order around $v^\sharp$. See the figure below. The graph now has two vertices, but still only one face.

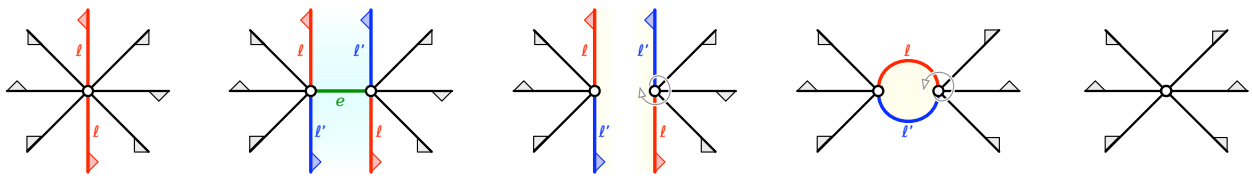
- Next we insert a parallel duplicate ℓ' of ℓ, also joining $v^\flat$ to $v^\sharp$, and splitting f into two faces, one of which has the boundary walk e, ℓ, e, ℓ'. The closure of this face in Σ is a Möbius band.

- Next we detach this Möbius band from Σ, resulting in a new surface Σ'. The only change in the rotation system is the deletion of the new edge e created in the first step. The resulting graph still has two vertices $v^\flat$ to $v^\sharp$ and two faces, one of which is a disk bounded by ℓ and ℓ'.

[2] Let $\phi : S^2 \setminus \{z\} \to \mathbb{R}^2$ be the standard stereographic projection map, where z is the 'north pole', and let $h\colon \Sigma \setminus \{v\} \to \mathbb{R}^2$. Then the function $\hbar\colon S^2 \to \Sigma$, defined by setting $\hbar(v) := z$ and $\hbar(x) := \phi^{-1}(h(x))$ for all $x \in \Sigma \setminus \{v\}$, is a homeomorphism.

- Finally, we return to a system of loops L' by contracting ℓ and deleting ℓ' (or vice versa).

The overall effect of these operations is *almost* the same as simply deleting ℓ from the rotation system. If the original system of loops L has signature $(\ell x \ell y)$, where x and y are strings of signed edge labels, then the new system of loops L' has signature $(x\overline{y})$, where $\overline{y}$ is the resulting of reversing y and changing the sign of every label.



Removing a 1-sided loop. (1) The initial system of loops. (2) Expanding e and inserting ℓ'; the closure of the shaded face is a Möbius band. (3) Detaching the Möbius band. (4) Redrawing the rotation system. (5) Contracting ℓ and deleting ℓ'.

By construction, Σ is homeomorphic to Σ' with one Möbius band attached. L' has one fewer edge than L, so by the induction hypothesis, $\Sigma' \cong \Sigma(g', h')$ for some non-negative integers g' and h'. Thus, $\Sigma \cong \Sigma(g', h'+1)$.

Finally, suppose L is non-empty and contains only two-sided loops. Let ℓ be an arbitrary loop in L. Let γ be another loop in L that alternates with ℓ in the rotation system. (There must be such a loop, because otherwise, the faces on ether side of ℓ would be distinct.) We again modify the graph and the surface in four stages.

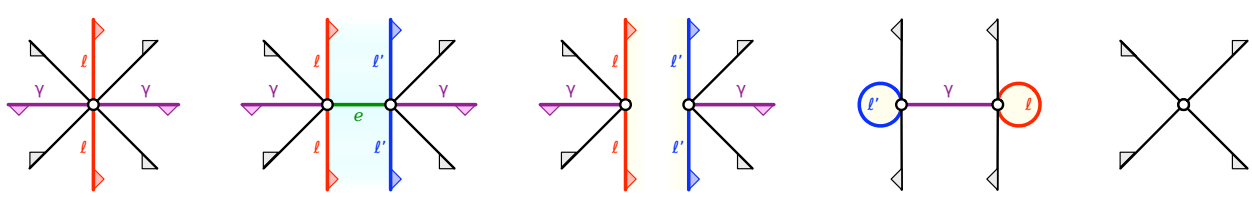
- Expand the vertex v into two vertices $v^\flat$ and $v^\sharp$ joined by an edge e. The loop ℓ remains an loop based at $v^\flat$; all the edges incident to v on one side of ℓ move to $v^\sharp$. In particular, γ is now an edge between $v^\flat$ and $v^\sharp$.

- Add a parallel loop ℓ' based at $v^\sharp$. The graph now has two faces, one bounded by the walk ℓ, e, ℓ, e'. The closure of this face is an annulus.

- Detach this annulus to get a new surface Σ', by deleting e from the rotation system. The resulting graph has two vertices and *three* faces. One of these faces is bounded entirely by ℓ; another by ℓ'. Because γ connects $v^\flat$ and $v^\sharp$, the graph is still connected.

- Finally, return to a system of loops L' by deleting both ℓ and ℓ' and contracting γ.

The overall effect of these operations is *almost* the same as simply deleting ℓ and γ from the rotation system. If the original system of loops L has signature $(\ell w \gamma x \ell y \gamma z)$, where w, x, y, z are strings of edge labels, then the new system of loops L' has signature $(zywx)$.



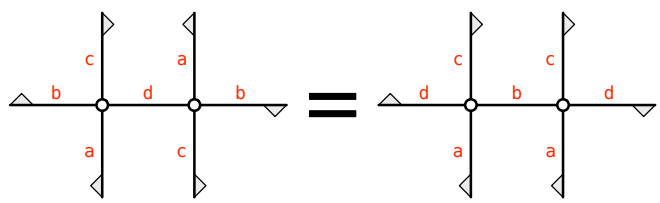
Removing a 2-sided loop. (1) The initial system of loops. (2) Expanding e and inserting ℓ'; the closure of the shaded face is an annulus. (3) Detaching the annulus. (4) Redrawing the rotation system. (5) Deleting ℓ and ℓ' and contracting γ.

By construction, Σ is homeomorphic to Σ' with one annulus attached. L' has two fewer edges than L, so by the induction hypothesis, $\Sigma' \cong \Sigma(g', h')$ for some non-negative integers g' and h'. Thus, $\Sigma \cong \Sigma(g'+1, h')$. $\square$

The following lemma, originally proved by Dyck [3], completes the proof of the classification theorem. Dyck's lemma can be stated less formally as follows: In the presence of another Möbius band, an annulus is equivalent to two Möbius bands.

Lemma 6.3. *If $h > 0$, then $\Sigma(g,h) \cong \Sigma(0,h+2g)$.*

Proof: Because handles and Möbius bands can be attached to the surface in any order, it suffices to prove that $\Sigma(1,1) \cong \Sigma(0,3)$. Consider the signed rotation system $(\mathtt{abcd})(\overline{\mathtt{abc}}\mathtt{d})$. Contracting edge $\mathtt{d}$ gives a system of loops with signature $(\mathtt{abcabc})$; given this system as input, the algorithm described in the proof of Theorem 6.2 produces $\Sigma(1,1)$. On the other hand, contracting edge $\mathtt{b}$ gives a system of loops with signature $(\mathtt{aadcc}\overline{\mathtt{d}})$; given this system of loops as input, the same algorithm produces $\Sigma(0,3)$. □



$\Sigma(1,1) \cong \Sigma(0,3)$

Corollary 6.4 (Surface Classification Theorem). *Every compact connected 2-manifold is homeomorphic to either $\Sigma(g,0)$ or $\Sigma(0,g)$, for some non-negative integer g.*

6.4 Oilers' Formula (ha ha)

The following theorem was first stated for convex polyhedra in $\mathbb{R}^3$ by Leonhard Euler (whose last name is pronounced 'oiler') [4], and then later generalized to polyhedra with non-trivial topology by Simon Antoine Jean l'Huilier (whose last name means 'the oiler') [7, 8].

The ***Euler characteristic*** $\boldsymbol{\chi(\Pi)}$ of a polygonal schema Π is the number of vertices minus the number of edges plus the number of faces: $\chi(\Pi) = V(\Pi) - E(\Pi) + F(\Pi)$. The Euler characteristic $\boldsymbol{\chi(\Sigma)}$ of a 2-manifold Σ is the Euler characteristic of any polygonal schema Π such that $\Sigma(\Pi) \cong \sigma$. The next theorem shows that $\chi(\Sigma)$ is well-defined.

Theorem 6.5. *Every polygonal schema for the surface $\Sigma(g,h)$ has Euler characteristic $2-2g-h$.*

Proof: Contracting an edge in a polygonal schema decreases both the number of vertices and the number of edges by 1, leaving the Euler characteristic unchanged. Similarly, deleting an edge decreases both the number of edges and the number of faces by 1, leaving the Euler characteristic unchanged. Thus, we can reduce Π to a system of loops L for the same surface, such that $\chi(L) = \chi(\Pi)$. The Euler characteristic of a system of loops is just 2 minus the number of loops.

Now consider the proof of Theorem 6.2. The only system of loops for the sphere is the empty system, which has Euler characteristic 2. Each time we remove a one-sided loop, we decrease the number of loops by 1, and therefore increase the Euler characteristic by 1. Each time we remove a two-sided loop, we decrease the number of loops by 2, and therefore increase the Euler characteristic by 2. The theorem now follows immediately by induction. □

Corollary 6.6. *Two compact connected 2-manifolds are homeomorphic if and only if (1) they are either both orientable or both non-orientable and (2) their Euler characteristics are equal.*

Corollary 6.7. *Let Π be a polygonal schema of complexity n. We can determine the homeomorphism class of $\Sigma(\Pi)$ in $O(n)$ time.*

6.5 Die Hauptvermutung

Theorem 6.8. *Any two polygonal schemata of the same 2-manifold have a common refinement.*

⊕⟹ ⟨⟨**Maybe next time.**⟩⟩

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