

CONSTRUCTIVE WHITNEY–GRAUSTEIN THEOREM: OR HOW TO UNTANGLE CLOSED PLANAR CURVES*

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Abstract. The classification of polygons is considered in which two polygons are *regularly equivalent* if one can be continuously transformed into the other such that for each intermediate polygon, no two adjacent edges overlap. A discrete analogue of the classic Whitney–Graustein theorem is proven by showing that the winding number of polygons is a complete invariant for this classification. Moreover, this proof is constructive in that for any pair of equivalent polygons, it produces some sequence of *regular transformations* taking one polygon to the other. Although this sequence has a quadratic number of transformations, it can be described and computed in real time.

Key words. polygons, computational algebraic topology, computational geometry, Whitney–Graustein theorem, winding number

AMS(MOS) subject classifications. 68Q20, 55M25

1. Why a circle differs from a figure-of-eight. First consider closed planar curves that are smooth. Intuitively, a “kink” on such a curve is a point without a unique tangent line. It seems obvious that there is no continuous deformation of figure-of-eight to a circle in which all the intermediate curves remain kink-free (see Fig. 1).

Figure 2 shows another curve that clearly has a kink-free deformation to a circle.

Let us make this precise. By a (closed planar) curve we mean a continuous function $C:[0, 1] \rightarrow E^2$, $C(0) = C(1)$, where E^2 is the Euclidean plane. The curve C is *regular* if the first derivative $C'(t)$ is defined and not equal to zero for all $t \in [0, 1]$, and $C'(0) = C'(1)$. Let $h:[0, 1] \times [0, 1] \rightarrow E^2$ be a *homotopy* between curves C_0 and C_1 , i.e., h is a continuous function and each $C_s:[0, 1] \rightarrow E^2$ ($0 \leq s \leq 1$) is a curve, where we define $C_s(t) = h(s, t)$ ($t \in [0, 1]$). The homotopy is *regular* if each C_s is regular. Two regular curves are *regularly equivalent* if there is a regular homotopy between them. A classical result known as the Whitney–Graustein theorem [5], [1] says that two curves are regularly equivalent if and only if they have the same winding number (up

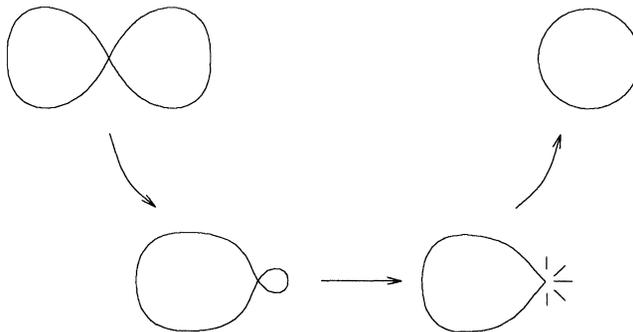


FIG. 1. Transforming a figure-of-eight to a circle: kink appears.

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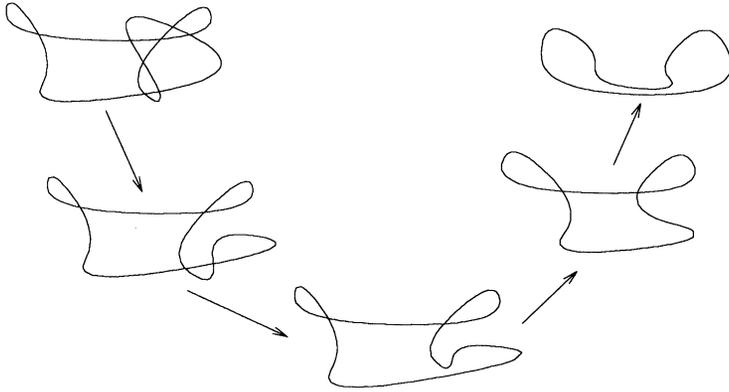


FIG. 2. A kink-free deformation.

to sign). Here, the winding number of a regular curve $C : [0, 1] \rightarrow E^2$ may be defined as follows. The “tangent map” of C is defined to be

$$\omega_C : S^1 \rightarrow S^1,$$

where S^1 is the unit circle and for any point $\theta = \theta(t) = \exp(2\pi t\sqrt{-1}) \in S^1$, $0 \leq t \leq 1$, we let $\omega_C(\theta) = C'(t)/|C'(t)|$. Note that ω_C is well defined since C is regular. Then the *winding number* of C is

$$\frac{1}{2\pi} \int_{S^1} d\omega_C.$$

The winding number is an integer. For instance, the winding number of the figure-of-eight is zero and the winding number of the circle is ± 1 (depending on the orientation of the curve C). Since these two curves have distinct winding numbers, the Whitney–Graustein theorem confirms our intuition that they are not regularly equivalent.

We should point out a closely related result of Hopf.¹ Hopf’s theorem [2] says that two maps $f, g : S^1 \rightarrow S^1$ are homotopic to each other if and only if they have the same winding number. In fact, Hopf’s theorem generalizes to higher dimensions for maps on the n -sphere S^n .

The purpose of this paper is to give a constructive version of the Whitney–Graustein theorem. The rest of this paper is organized as follows. In § 2, we formulate the discrete (polygonal) version of regular curves and regular equivalence. Section 3 introduces a normal form for polygons. In § 4, we prove the Whitney–Graustein theorem for polygons. An algorithm is developed in § 5 using the insights from the proof. Section 6 concludes the paper.

2. Classification of regular polygons. Computational issues arising from the Whitney–Graustein theorem include asking for a procedure to decide equivalence of two given regular curves and to construct a regular homotopy between two equivalent curves. To obtain computational complexity results, we discretize these questions. The natural candidates for discretized regular curves would be polygons (i.e., closed polygonal paths). Unfortunately, polygons are never “regular” since they automatically

¹ In our original paper [3], we mistakenly referred to the Whitney–Graustein theorem as Hopf’s theorem. We are grateful to Gert Vegter for setting the record straight.

have kinks at their vertices. This would seem to destroy any hope for a Whitney-Graustein theorem for polygons. It turns out that we can isolate the essential features of the original “regularity” assumption, and transfer these features into the polygonal setting. So there are “regular” polygons after all.

DEFINITION. A *path* Π is specified by a sequence

$$\Pi = \langle v_1, v_2, \dots, v_n \rangle, \quad n \geq 2$$

of points which we call *vertices*. The *initial* and *final* vertices are v_1 and v_n , respectively, and if $n \geq 3$, we call v_2, \dots, v_{n-1} the *interior vertices*. We require adjacent vertices to be distinct: $v_i \neq v_{i+1}$ for $i = 1, \dots, n-1$. The edges of the path are the line segments $[v_i, v_{i+1}]$ for $i = 1, \dots, n-1$. The *reverse* of $\langle v_1, v_2, \dots, v_n \rangle$ is $\langle v_n, \dots, v_2, v_1 \rangle$. A *closed path* is one of the form

$$\Pi = \langle v_1, \dots, v_n, v_{n+1} \rangle, \quad n \geq 2$$

such that $v_{n+1} = v_1$. Intuitively, if we identify the first and last vertices of a closed path, then we would like to consider two closed paths as equivalent if one sequence can be obtained from the other by a cyclic shift. More precisely, two closed paths Π, Π' are *cyclically equivalent* if $\Pi = \langle v_1, \dots, v_n, v_1 \rangle$ and Π' is equal to

$$\langle v_i, v_{i+1}, \dots, v_n, v_1, v_2, \dots, v_{i-1}, v_i \rangle$$

for some $i = 1, \dots, n$. An *oriented polygon* P on $n \geq 2$ vertices is defined as the cyclic equivalence class of some closed path $\langle v_1, \dots, v_n, v_1 \rangle$. The *reverse* of an oriented polygon is defined as expected. A *nonoriented polygon* P is the class of closed paths cyclically equivalent to some closed path or its reverse. For short, “polygon” is understood to mean nonoriented polygon. If P is the polygon consisting of closed paths cyclically equivalent to $\langle v_1, \dots, v_n, v_1 \rangle$ or its reverse, we denote P by

$$P = (v_1, \dots, v_n).$$

So P could also be written as $(v_n, v_{n-1}, \dots, v_1)$, and also $(v_2, v_3, \dots, v_n, v_1)$, etc.

DEFINITION. Let v_i be a vertex of a path $\langle v_1, \dots, v_n \rangle$ or a polygon (v_1, \dots, v_n) , where v_i is an interior vertex in case of the path. Then v_i is a *kink* if the two edges $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$ incident on v_i overlap. By definition, the initial and final vertices of a path are never kinks, and the vertices of a two-vertex polygon are always kinks. A path or polygon is *regular* if it has no kinks. An oriented polygon is *regular* if its nonoriented counterpart is regular.

Here, as throughout the paper, arithmetic on subscripts of vertices of a polygon P is modulo n , the number of vertices of P . Note that regularity precludes neither nonadjacent vertices from coinciding nor nonadjacent edges from overlapping.

Let v_i be an interior vertex of a regular path $\langle v_1, \dots, v_n \rangle$. Then the *turning angle* at v_i is defined to be the angle of absolute value less than π that is equal to $\theta_i - \theta_{i-1} \pmod{2\pi}$, where θ_i is the orientation of the ray from v_i through v_{i+1} . Note that if v_i were a kink, we would have an ambiguous choice of either π or $-\pi$. We say the path makes a *right-turn*, *left-turn*, *no-turn* at v_i according as the turning angle at v_i is negative, positive, or zero, respectively. Let P be a regular oriented polygon that is cyclically equivalent to the closed path $\Pi = \langle v_1, v_2, \dots, v_n, v_1 \rangle$. We define the *turning angle* of a vertex v_i of P as follows. If $i \neq 1$, then this is equal to the turning angle of v_i when v_i is regarded as a vertex of Π ; if $i = 1$, then this is the angle of absolute value less than π that is equal to $\theta_1 - \theta_n \pmod{2\pi}$. The *winding number* of an oriented regular polygon is the sum of the turning angles at each of its vertices, divided by 2π . We see that the winding number of a regular oriented polygon is an integer which is equal to

the negative of the winding number of the reverse oriented polygon. Hence we may define the *winding number* of a nonoriented polygon to be equal to the absolute value of the winding number of any one of its two oriented versions.

It is capturing this notion of turning angle that necessitates our regularity requirement. Henceforth, we assume all polygons and paths are regular unless otherwise noted.

We introduce three types of *regular transformations* of a polygon $P = (v_1, \dots, v_n)$. Let $i = 1, \dots, n$.

(T0) Insertion. We may transform P to

$$Q = (v_1, \dots, v_i, u, v_{i+1}, \dots, v_n),$$

where u is a point in the relative interior of the edge $[v_i, v_{i+1}]$.

(T1) Deletion. We may transform P to

$$Q = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$

provided v_{i-1}, v_i, v_{i+1} are collinear (and hence, by regularity of P , v_i lies strictly between v_{i-1} and v_{i+1}).

(T2) Translation. We may transform P to

$$Q = (v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_n),$$

where u is any point such that for all $0 \leq t \leq 1$, the polygon $Q_t = (v_1, \dots, v_{i-1}, (1-t)v_i + tv_i, v_{i+1}, \dots, v_n)$ is regular. In particular, $Q = Q_1$ is regular.

It is not hard to characterize the possible choices for the point u in (T2). Relative to vertex v_i , we define two *forbidden cones* (at v_{i-1} and at v_{i+1} , respectively): the forbidden cone at v_{i-1} is bounded by the two rays emanating from v_{i-1} , one ray directed towards v_{i-2} and the other directed away from v_{i+1} . See Fig. 3. Of the two complementary cones bounded by these rays, we choose the one that does not contain v_i . The forbidden cone at v_{i+1} is similarly defined, being bounded by the two rays emanating from v_{i+1} (one directed towards v_{i+2} and the other directed away from v_{i-1}). Each cone is a closed region so it includes the bounding rays. In (T2), we are free to choose any u as long as u is not in the union of the two forbidden cones.

DEFINITION. We say that two polygons P, Q are (*regularly*) *equivalent* if one can be transformed to the other by a finite sequence of regular transformations.

Clearly, insertions and deletions are inverse operations but translations are “self-inverses.” It is easily seen that regular equivalence is an “equivalence” relation in the

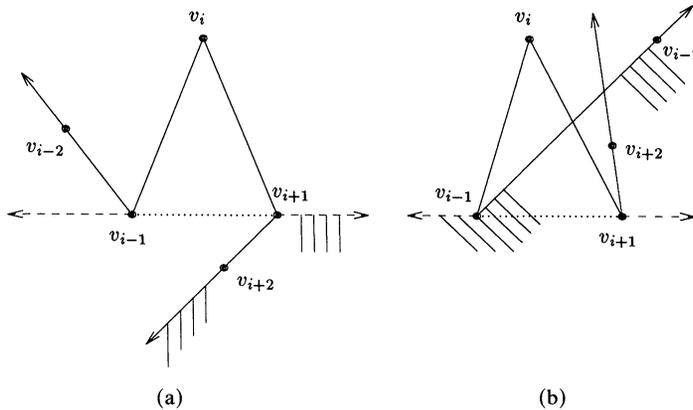


FIG. 3. (a) Forbidden cone at v_{i+1} . (b) Forbidden cone at v_{i-1} . Forbidden cones are shaded.

usual mathematical sense. Also, our transformations preserve winding number. We would expect the discrete analogue of the Whitney–Graustein theorem to assert that winding number is a complete invariant for regular equivalence among polygons. Towards this end, we will define a “normal form polygon” for each winding number and show that a quadratic number of regular transformation steps suffices to bring any polygon to one of these normal forms. Our algorithm finds these quadratically many steps in linear time (*sic*). (The subtitle of this paper refers to this transformation sequence from P to its normal form \hat{P} as “untangling.”) This is sufficient to solve the problem of finding the regular transformations from any polygon P to any other equivalent polygon Q : first transform P to its normal form \hat{P} and then apply *in reverse order* the inverse of each of the transformation steps that takes Q to its normal form $\hat{Q}(=\hat{P})$.

To think about what normal forms might be desirable, we note (see Fig. 4) that the triangle and the bow-tie are obvious candidates for normal forms. Perhaps less convincingly, the 5-point star (5-star) also seems like a good candidate for a normal form.

Figure 5 illustrates a sequence of regular transformations (some steps are omitted) from the “Victoria Cross” to the 7-star polygon (with one fewer vertex). It does not seem obvious how we can systematically transform the Victoria Cross to the 7-star polygon, even if we were told that such a sequence of transformations exist.

Remark. Any smooth closed curve can be approximated by a polygon, and any homotopy between smooth curves can be discretely approximated by a series of our (T0), (T1), (T2) transformations. So in some sense, we have solved the original question of constructing regular homotopies between equivalent regular curves.

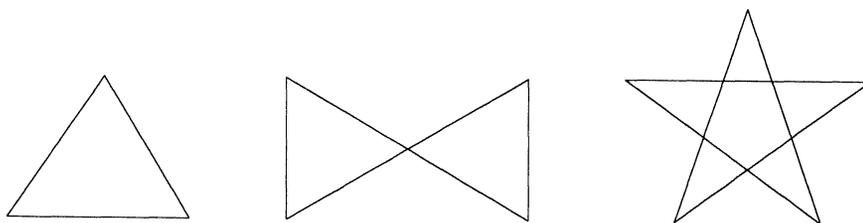


FIG. 4. The triangle, bow-tie, and 5-star.

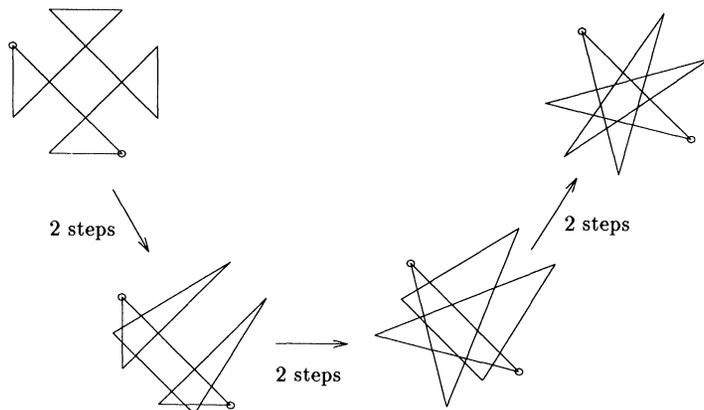


FIG. 5. Reduction of the Victoria Cross.

3. Star polygons.

DEFINITION. A polygon is *reducible* if it is equivalent to one with fewer vertices. It is *irreducible* otherwise.

We can easily check that the triangle, bow-tie, and 5-star cannot be transformed by (T1) or (T2) transformations into any polygon with fewer vertices. But it turns out that even allowing (T0) transformations (which insert new vertices), these polygons cannot be transformed into ones with fewer vertices; in other words, they are irreducible. Since they have different numbers of vertices, it follows that they are inequivalent to each other. In fact, any polygon with at most six vertices is equivalent to one with three candidates. In particular, there is no irreducible polygon on six vertices.

The 5-star is equivalent to the polygons in Fig. 6. The first polygon (the Fox) in Fig. 6 has the minimum number of self-intersections among its equivalence class, so one could argue that the Fox is a better choice for normal form than the 5-star.

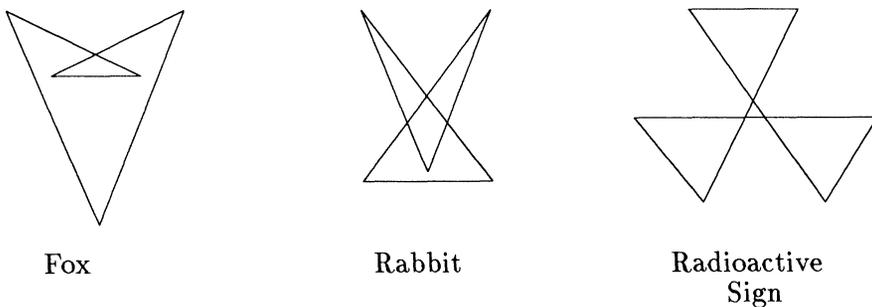


FIG. 6. Equivalent to the 5-star: a Fox, a Rabbit, and a Radioactive Sign.

LEMMA 1. Every polygon P can be transformed by (T2) transformations into a polygon Q , all of whose vertices are distinct and lie on a circle C . Here C is any circle that contains P in its interior.

Proof. Let C be such a circle. Recall that for each vertex v of P , we have defined two “forbidden cones.” The “nonforbidden region” of v is the complement of the union of these two cones. It is not hard to see that the nonforbidden region of v contains a nonempty open cone K of infinite rays emanating from v . Any ray R from this cone K intersects the circle C at some point u , and a translation will take v to u . Since K is open, we can choose R to ensure that u is distinct from each vertex of P already on C . This can be repeated for successive vertices v of P . \square

Henceforth we assume that all vertices of polygons and paths are distinct (with the obvious exclusion for closed paths) and lie on some circle. The circle depends on the individual polygon or path.

DEFINITION. A path $\Pi = \langle v_1, \dots, v_n \rangle$ is called a *star path* if each edge $e_i = [v_i, v_{i+1}]$ (for $i = 1, \dots, n - 1$) intersects each of the edges

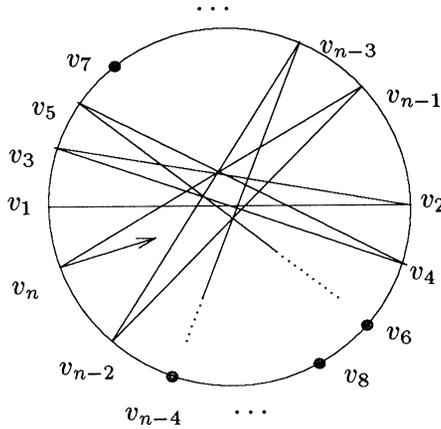
$$e_1, e_2, \dots, e_{i-1}.$$

See Fig. 7. Since edges are closed line segments, e_i ($i \geq 2$) always intersects e_{i-1} . A polygon $P = (v_1, \dots, v_n)$ is called an n -star if for some choice of an *initial* vertex v_i , $i = 1, \dots, n$, the path

$$(1) \quad \Pi_i = \langle v_i, v_{i+1}, \dots, v_n, v_1, v_2, \dots, v_{i-1} \rangle$$

is a star path.

This terminology agrees with what we have called a 5-star. A triangle is a 3-star and a bow-tie a 4-star. Figure 8 shows the next few n -stars.



(n even)

FIG. 7. Star path.

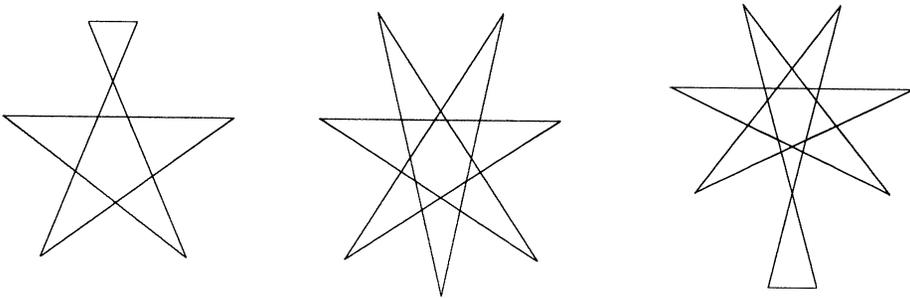


FIG. 8. 6-, 7-, and 8-stars.

Let us make some simple observations about star paths. A path is *right-turning* (respectively, *left-turning*) if it makes only right-turns (respectively, left-turns) at each of its interior vertices. A path is right-turning if and only if its reverse is left-turning. We see that a star path of length $n \geq 3$ is either left-turning or right-turning. Let $\Pi_1 = \langle v_1, \dots, v_n \rangle$ be a star path. For each $i = 1, \dots, n$, let Π_i be the path in (1). Π_i is basically some cyclic shift of Π_1 . If n is odd, then Π_i is a star path for each i . For even values of $n \neq 4$, Π_i is not a star path for all $i > 1$. For $n = 4$, Π_1 and Π_3 are the only star paths.

Notation. If u_1, \dots, u_k , $k \geq 3$, are distinct vertices of polygon P lying on some circle C , we write

$$(2) \quad u_1 < u_2 < \dots < u_k$$

to mean that, as we traverse the circle C in a clockwise fashion starting from u_1 , we will meet the vertices u_1, u_2, \dots, u_k in this order (though other vertices not among the u 's may intervene). Thus $u_1 < u_2 < \dots < u_k$ is equivalent to $u_2 < u_3 < \dots < u_k < u_1$, etc. We call a list of the form (2) a *cyclic permutation* on the vertices u_1, \dots, u_k . So any polygon or path whose vertices include u_1, \dots, u_k induces a cyclic permutation on u_1, \dots, u_k . Note that the cyclic permutation induced by a path or its reverse are the same.

For example, $\Pi = \langle v_1, \dots, v_n \rangle$ is a left-turning star path if and only if Π induces the following cyclic permutation on the points v_1, \dots, v_n :

$$(3) \quad v_1 < v_3 < v_5 < \dots < v_{\text{odd}} < v_2 < v_4 < v_6 < \dots < v_{\text{even}}$$

where $v_{\text{odd}} = v_n, v_{\text{even}} = v_{n-1}$ if n is odd, and $v_{\text{odd}} = v_{n-1}, v_{\text{even}} = v_n$ if n is even. Figure 7 illustrates the case where n is even.

The following gives a method for moving a vertex to another given position.

LEMMA 2 (moving a vertex). *Let $P = (u_1, \dots, u_n)$ be a polygon whose vertices lie on a circle C . Let x be any point of C . Let $A \subseteq C$ be one of the two closed arcs bounded by u_1 and x . Call an edge of P active if at least one of its endpoints is in A . The vertices in A are thus partitioned into connected components where two vertices are “connected” if they are joined by a path of active edges. Let U be a union of any number of connected components with the provision that $u_1 \in U$ and (if x is a vertex of P) $x \in U$. Then for any open neighborhood N of x , there is a sequence of $|U|$ translations that moves u_1 to x , moves the vertices in $U - \{u_1\}$ into $(C \cap N) - A$, and keeps all the remaining vertices of P fixed. Moreover the relative ordering of the elements in U is preserved.*

Proof. The vertices in U can be ordered according to increasing distance from x , where distance is measured along the arc A . See Fig. 9. Starting from the vertex in U that is closest to x , we translate each one in turn into $(C \cap N) - A$. The last vertex to be translated would be u_1 itself, and this can be translated to x . To see why this works, note that a vertex u_j can be translated to any position in C provided the induced cyclic permutation on $u_{j-2}, u_{j-1}, u_j, u_{j+1}, u_{j+2}$ is preserved. So when we try to translate $u_j \in U$ as described, we see that each of $u_{j-2}, u_{j-1}, u_{j+1}, u_{j+2}$ either has been moved into $(C \cap N) - A$ already or lies outside of A . In any case, we can translate u_j to some position in $(C \cap N) - A$ which preserves the induced permutation of U . \square

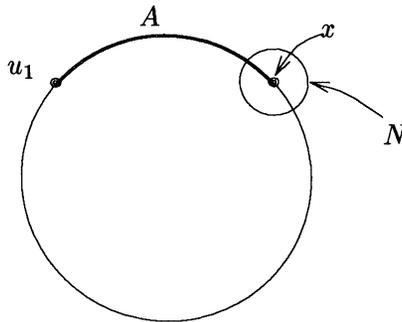


FIG. 9. Moving clockwise from u_1 to x .

The transformation described by this lemma may be described as “moving u_1 to x relative to (A, U, N) .” The smallest choice for U is the union of the connected component of u_1 with the connected component of x (regard the latter component to be empty if x is not a vertex). The largest choice for U is the set of all the vertices in A . If N is chosen small enough so that N contains no vertices of P , then we refer to the move corresponding to the smallest choice for U as a *weak move clockwise/ counterclockwise from u_1 to x* (where A is the arc clockwise/ counterclockwise from u_1 to x). If U is the largest possible choice, it is similarly known as a *strong move*. Note that a strong move preserves the induced cyclic permutation of P .

LEMMA 3. *Let $P = (u_1, \dots, u_n)$ and $A = (v_1, \dots, v_n)$ be two polygons with all their vertices lying on a common circle C . Let σ and τ be the cyclic permutations on the u 's*

and v 's induced by P and Q , respectively. If σ and τ are similar (in the sense that after renaming each u_i by v_i , they are identical), then P and Q are regularly equivalent in less than or equal to $1.5n$ steps.

Proof. As usual, the u 's are pairwise distinct and so are the v 's; without loss of generality, assume σ and τ are both the identity. Using the above lemma, we make a strong move clockwise or counterclockwise from u_1 to v_1 ; of the two possible clock directions, we could choose the one using at most $n/2$ translational steps. Applying the lemma again, we make a strong move from either u_2 to v_2 or v_2 to u_2 using only one translational step. In general, assuming u_1, \dots, u_{i-1} is coincident with v_1, \dots, v_{i-1} , we can move either u_i to v_i or v_i to u_i in only one step. \square

For instance, any n -star (v_1, \dots, v_n) induces the cyclic permutation (3). Hence this lemma tells us that any two n -stars (which we may assume to lie on a common circle C) are regularly equivalent. In view of this, we henceforth refer to "the n -stars" as if these were unique for each n .

LEMMA 4. *Let n, m be odd positive integers or equal to four. If $n \neq m$, then the n -star and the m -star are inequivalent.*

Proof. We check that the winding number of the 4-star is zero and for each positive integer k , the $(2k+1)$ -star has winding number k . The result then follows from the fact that the winding number of a polygon is unchanged by any regular transformation. \square

This lemma supplies us with an infinite list of inequivalent polygons. We will prove that every regular equivalence class is represented in this list.

4. The Whitney–Graustein theorem for polygons. The main result of this section is Theorem 5.

THEOREM 5 (canonical form). *Every polygon can be transformed by a sequence of (T1) and (T2) transformations into an n -star, for some n that is either odd or equal to four.*

COROLLARY 6. *An n -star is irreducible if and only if $n = 4$ or n is odd.*

Proof. Suppose that an n -star is irreducible. Then the theorem implies that n must be four or odd. Conversely, let $n = 4$ or odd. If an n -star were reducible, then the theorem shows that it would be reducible to an m -star for some $m < n$ where $m = 4$ or odd. This contradicts the previous lemma that the n - and m -stars are inequivalent. \square

We prove the canonical form theorem by a sequence of lemmas.

A polygon that can (respectively, cannot) be transformed to one with fewer vertices using just (T1) and (T2) transformations will be called *semireducible* (respectively, *semi-irreducible*). (Of course, in view of Theorem 5, semireducibility turns out to be the same concept as reducibility.)

Notation. For compactness, we will usually write only the *indices* (i.e., subscripts) of vertices in place of the vertices themselves. Thus we write $P = (1, 2, \dots, n)$ for a polygon on n vertices. Combined with an earlier notation, we may write " $1 < 3 < 2$ " to mean " $v_1 < v_3 < v_2$ ": this is hopefully not too confusing.

The following simple fact is often used.

LEMMA 7 (deleting a vertex). *Suppose $P = (1, \dots, n)$ ($n \geq 5$) is such that the pair of edges $[1, 2]$ and $[3, 4]$ does not intersect, and also the pair $[2, 3]$ and $[4, 5]$ does not intersect. Then P is equivalent to $(1, 2, 4, 5, \dots, n)$ after a (T2) followed by a (T1) transformation. In other words, we may delete index 3.*

Proof. See Fig. 10. The nonintersection assumptions of the lemma imply that the interior of the triangle $\Delta 234$ is nonforbidden for vertex 3. Hence we can translate vertex

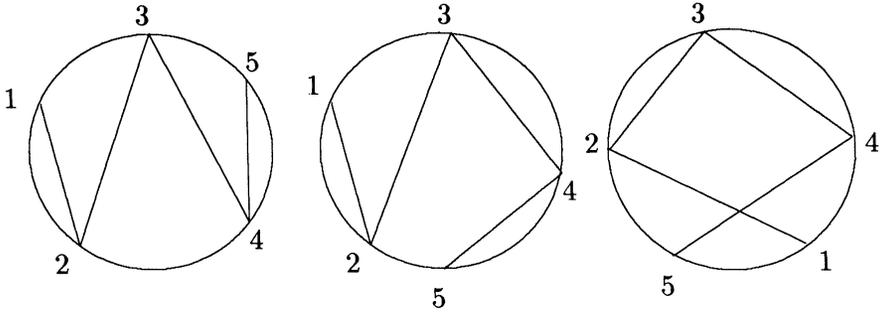


FIG. 10. Can delete index 3.

3 to the midpoint of edge $[2, 4]$ by a (T2) transformation. Next, a (T1) transformation eliminates vertex 3. \square

Henceforth, whenever we delete vertices, it is by appeal (usually implicit) to this lemma.

We say that $P = (1, 2, \dots, n)$ contains an *N-shape* if $n \geq 4$ and for some choice of index i , we have

$$i < i+1 < i+3 < i+2$$

(Fig. 11a) or

$$i < i+2 < i+3 < i+1$$

(Fig. 11b). We call $(i, i+1, i+2, i+3)$ an *N-shape*.

LEMMA 8. A semi-irreducible polygon $P = (1, 2, \dots, n)$ does not contain an *N-shape* unless $n = 4$.

Proof. By way of contradiction, assume that P has an *N-shape*. By symmetry, assume that $1 < 2 < 4 < 3$ (Fig. 12).

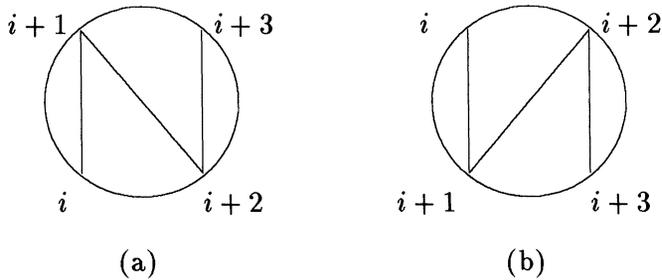


FIG. 11. An *N-shape*.

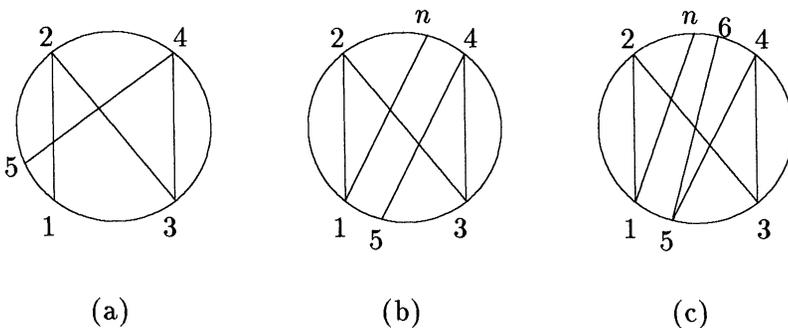


FIG. 12. Reduction of an *N-shape*.

The result is true for $n = 4$, so suppose $n \geq 5$. Since P is semi-irreducible, by the previous lemma, the edge $[4, 5]$ must intersect $[2, 3]$; hence $3 < 5 < 2$. Similarly, $2 < n < 3$. This shows that $n \neq 5$ so assume $n \geq 6$. If $1 < 5 < 2$ (Fig. 12(a)) then we can translate index 2 so that $1 < 2 < 5$ (this translation can occur because $2 < n < 3$). Then we can delete 3, a contradiction. Therefore, we have $3 < 5 < 1$. By symmetry, we have $2 < n < 4$. The situation is shown in Fig. 12(b).

If $n = 6$ then it is easy to see that P is semireducible. Otherwise, consider the location of index 6. There are two cases. First suppose $n < 6 < 4$ (Fig. 12(c)). If index 7 is such that $5 < 7 < 1$ then we can delete index 5, a contradiction. Otherwise we may translate index 5 so that $1 < 5 < 2$, which reduces to a previous case (Fig. 12(a)). In the second case, $6 < n < 4$ and we can translate index 4 so that $4 < n < 3$ (and hence $4 < n < 2$). This again reduces to a previous case. \square

Note that a polygon does not contain any N-shape if and only if its oriented versions are left-turning or right-turning.

COROLLARY 9. *An n -star is semireducible if n is even and not equal to four.*

Proof. If $\langle 1, \dots, n \rangle$ is a star path and n is even, then $(n, 1, 2, 3)$ forms an N-shape. Since $n \neq 4$, the previous lemma implies that $(1, \dots, n)$ is semireducible. \square

We say that $P = (1, 2, \dots, n)$ contains a U-shape if $n \geq 4$ and for some choice of index i , we have

$$i < i+1 < i+2 < i+3$$

(Fig. 13(a)) or

$$i < i+3 < i+2 < i+1$$

(Fig. 13(b)). We call $(i, i+1, i+2, i+3)$ a U-shape.

LEMMA 10. *A semi-irreducible polygon $P = (1, 2, \dots, n)$ cannot contain a U-shape.*

Proof. We can easily check this for $n = 3, 4$, and 5; so assume that $n \geq 6$. Suppose indices $(2, 3, 4, 5)$ form a U-shape as in Fig. 14, $2 < 5 < 4 < 3$.

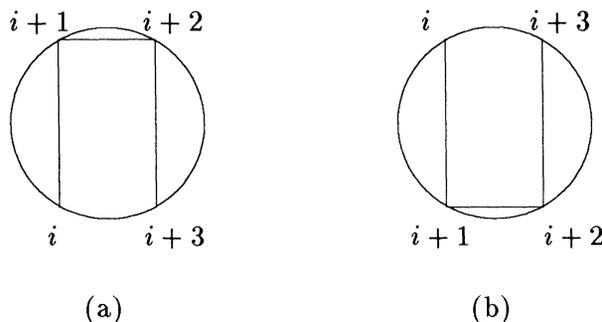


FIG. 13. A U-shape.

Since index 3 cannot be deleted, we must have $4 < 1 < 3$; similarly, since index 4 cannot be deleted, $4 < 6 < 3$. Suppose that the relative positions of indices 1 and 6 satisfy

$$(4) \quad 4 < 1 < 6 < 3,$$

as in Fig. 14(a). Then we may translate index 3 so that $1 < 3 < 6$ and then delete index 4, a contradiction. Hence we may assume the situation of Fig. 14(b), with $4 < 6 < 1 < 3$.

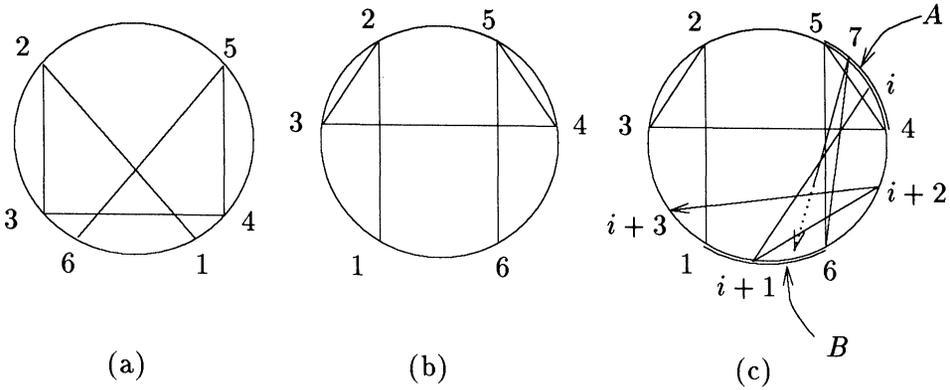


FIG 14. The elimination of U-shape.

Note that the path $\langle 1, \dots, n, 1 \rangle$ is left-turning since P contains no N-shape. This means $i < i - 1 < i + 1$ holds for all i . In particular, $1 < n < 2$ and $5 < 7 < 6$. This implies $n \geq 8$.

Let A be the arc clockwise from index 5 to index 4, including the index 5 but not 4. Similarly, B is clockwise from index 6 to index 1, including 6 but not 1. First note that we can translate index 7 into A . If index 8 is not in B then we can translate index 6 so that (4) holds, a contradiction. So index 8 lies in B and $n \geq 9$. But $7 < 9 < 8$ implies $n \geq 10$.

Suppose inductively that we have shown for some odd $i \geq 7$ (a) the indices $5 < 7 < \dots < i - 2 < i$ all lie in A , (b) the indices $6 < 8 < \dots < i - 1 < i + 1$ all lie in B , and (c) $n \geq i + 3$ (see Fig. 14(c)).

We claim that either we can extend the inductive assumptions (a)-(c) or we get a contradiction. First we can translate $i + 2$ into A so as to extend (a). Next, if $i + 3$ does not lie in B then let x be a point such that $1 < x < 3$ and $1 < x < i + 3$. We make a weak move from index 6 clockwise to x (see Lemma 2). Note that neither of the edges $[3, 4]$ and $[i + 2, i + 3]$ is active relative to the arc clockwise from 6 to x , and hence index 1 is fixed by the weak move. But now we are in the situation of (4) again, a contradiction. So assume that $i + 3$ is in B . It is easy to see from this that we have extended assumption (b). Now $i + 3$ is in B implies $n \geq i + 4$. But since $i + 2 < i + 4 < 1$, we see that in fact $n \geq i + 5$, which is assumption (c) extended.

Since we cannot extend the inductive assumptions indefinitely, we will eventually derive a contradiction. \square

We give one more lemma before proving the main result of this section.

LEMMA 11. Let $P = \langle 1, 2, \dots, n \rangle$, $n \geq 5$, be any semi-irreducible polygon. Then $\Pi = \langle 1, 2, \dots, n \rangle$ is a star path.

Proof. We now know that P contains no N- and no U-shapes. We will show that if $\Pi_v = \langle 1, 2, \dots, v \rangle$ (for $v = 3, \dots, n - 1$) is a star path, then Π_{v+1} is a star path. Consider the situation in Fig. 15 (without loss of generality, assume $1 < 3 < 2$).

The induction basis $v = 3$ follows from the previous two lemmas since if Π_4 is not a star path, then it forms a U- or an N-shape. The same remark holds for $v = 4$, so let $v \geq 5$.

First suppose v is odd. If Π_{v+1} is not a star path then $1 < v + 1 < v - 2$. Let x be a point such that $1 < x < 3$ and $1 < x < v + 1$. We do a weak move clockwise from 4 to x . Note the edges $[2, 3]$ and $[v, v + 1]$ are not active. Hence index 1 is left fixed by the weak move. So $\langle 1, 2, 3, 4 \rangle$ is now a U-shape, a contradiction.

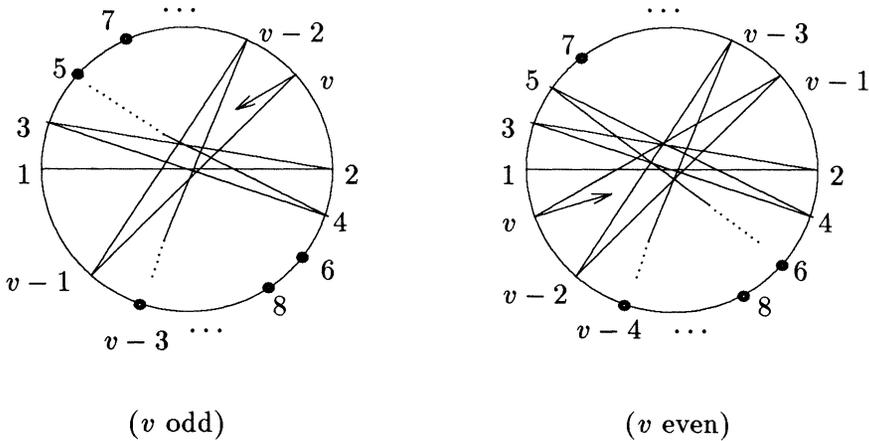


FIG. 15. Star paths from semireducible polygons.

Suppose v is even. If Π_{v+1} is not a star path then $2 < v+1 < v-2$. Let x be such that $2 < x < v+1$ and A be the arc from $v-2$ counterclockwise to x . We do a weak move from $v-2$ to x . Note that edges $[1, 2]$ and $[v-1, v]$ are not active. Hence $v+1$ is left fixed. So $(v-2, v-1, v, v+1)$ is now a U-shape, a contradiction. \square

The main result, Theorem 5, now follows: given any polygon P , we can reduce it by (T1) and (T2) transformations until it is semi-irreducible. Then, by the last lemma, the result must be an n -star. By Corollary 9, n is odd or equal to four. This proves the theorem.

Since there is a unique canonical form for each winding number and conversely, this proves that the winding number is a complete invariant for regular equivalence. This is the polygonal version of the Whitney-Graustein theorem.

5. Algorithm. The proof of the canonical form theorem contains an implicit quadratic-time algorithm to transform a polygon to its normal form. We now give a *real-time* algorithm to construct a similar sequence of quadratic transformation steps—this apparent paradox is soon clarified.

The algorithm *processes* the input vertices in order. In the generic situation, the vertices v_1, \dots, v_{j-1} have been processed and the polygon has been transformed into an equivalent polygon

$$P_j = (u_1, \dots, u_{i-1}, v_j, \dots, v_n).$$

Furthermore, we assume that

$$\langle u_1, \dots, u_{i-1} \rangle$$

forms a star path. (Throughout this section, the indices i and j will have this special meaning.) The *current vertex* being processed is v_j although our algorithm may look ahead slightly, up to v_{j+3} . (The algorithm halts after “wrapping around” to process $v_{n+2} = v_2$.)

An interesting feature of the algorithm is that it uses only $O(1)$ runtime memory. More precisely, at the moment of processing v_j , we only need in the active memory (i) the values of the indices i, j ; (ii) the values of the vertices

$$v_j, v_{j+1}, v_{j+2}, v_{j+3};$$

and (iii) the *sign information*: whether the paths

$$\langle u_{i-2}, u_{i-1}, v_j \rangle \quad \text{and} \quad \langle u_{i-1}, v_j, v_{j+1} \rangle$$

make right or left turns at their respective interior vertices. The transformed vertices u_1, \dots, u_{i-1} are output in an *even stack* and *odd stack*, storing the odd and even indices, respectively.

Our algorithm runs in real time in the sense that each vertex takes $O(1)$ time to process. However, we may output for vertex v_j a sequence of $O(j)$ transformation steps. To understand how $O(j)$ steps can be described in $O(1)$ time, recall the “weak move” subroutine from Lemma 2. It turns out that whenever we output an unbounded number of transformation steps, it is through making one or two such weak moves. The vertices to be translated in such a weak move turn out to form a contiguous block of elements in the odd or even stack, and so the weak move has a constant size description (of a suitable sort). After processing all n vertices, we produce $O(n^2)$ transformation steps overall. Thus P can be transformed to an irreducible star polygon in $O(n^2)$ transformation steps.

For simplicity, we will not explicitly specify the (T1), (T2) transformation steps to output. But each step of the algorithm will be given justification and the reader can easily deduce the transformation steps needed.

Since u_1, \dots, u_{i-1} are obtained by transformations of v_1, \dots, v_{j-1} , and since we may delete but never insert vertices, the relation

$$i \leq j$$

always holds. Therefore it is unambiguous to refer to the vertices by their indices: an index k refers to v_k if $k < i$ and refers to u_k if $k \geq j$. We assume that $j \geq 4$. To initialize, we may let $u_i = v_i$ for $i = 1, 2, 3$ and $j = 4$. Without loss of generality, assume that the path $\langle 1, 2, \dots, i-1 \rangle$ is left-turning.

There are four cases to consider while processing vertex v_j : the triple $(i-2, i-1, j)$ (i.e., (u_{i-2}, u_{i-1}, v_j)) represent either a left turn or a right turn, and i is either odd or even. First assume that i is odd (see Fig. A).

Case A. $(i-2, i-1, j)$ is a right turn.

Case A1. $(i-1, j, j+1)$ is a left turn. We pop the even stack which contains index $i-1$. In effect, we have decremented i by one.

Justification. See Fig. B. We may delete index $i-1$. To reconstruct the sign information, note that the turns $(i-3, i-2, j)$ and $(i-2, j, j+1)$ are both left turns. (As illustrated here, we could always reconstruct such sign information without

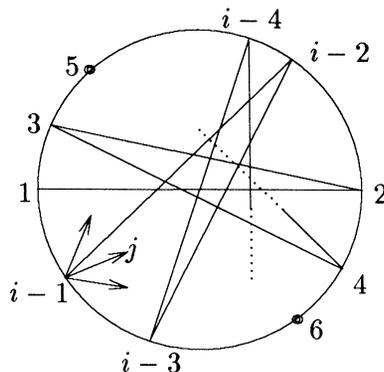


FIG. A. Case $i = \text{odd}$.

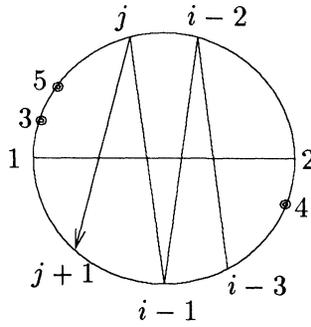


FIG. B. Case A1.

reexamining the vertices $i-1$, $i-2$, or $i-3$. Henceforth we will not explicitly mention this.)

Case A2. $(i-1, j, j+1)$ is a right turn. We have three subcases depending on the relative positions of indices j and $j+1$.

Case A21. $j < j+1 < 2$. After at most one translation, we may delete index $i-1$. In effect we decrement i by one.

Justification. See Fig. C. By translating index $i-2$ to some neighborhood of index 2 if necessary, we may assume that $j < j+1 < i-2$. Now delete index $i-1$.

Case A22. $j < 2 < j+1 < 1$. After $O(j)$ translations, we may delete index $i-2$. Effectively we decrement i by one.

Justification. See Fig. D. By translating index $i-2$ to some neighborhood of index 2, and by making a weak move (Lemma 2) counterclockwise from index $i-4$ to some neighborhood of index 1, we may assume that $i-4 < j < i-2$. As noted, this weak move may involve $O(j)$ translational steps but this has a constant size description. Then we are in the situation of Fig. D(a). We may now translate index $i-1$ clockwise until $i-4 < i-1 < j$ (Fig. D(b)), and then delete $i-2$. Now $(1, \dots, i-3, i-1)$ is a star path.

Case A23. $j < 1 < 2 < j+1$ (see Fig. E). We consider two subcases for index $j+2$.

Case A231. $j < 1 \leq j+2$. By translating index j clockwise until $1 < j < 2$, we reduce this to Case A22.

Case A232. $j < j+2 < 1$. See Fig. F. Next consider subcases depending on index $j+3$.

Case A2321. $(j+1, j+2, j+3)$ is a right turn. Increment j by two.

Justification. We may delete $j+1$ and j (in that order).

Case A2322. $(j+1, j+2, j+3)$ is a left turn. See Fig. G(a). After some translations, we replace index $i-1$ by $j+2$; effectively we increment j by three.

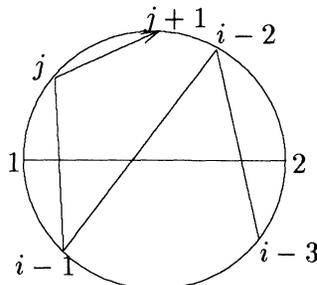


FIG. C. Case A21.

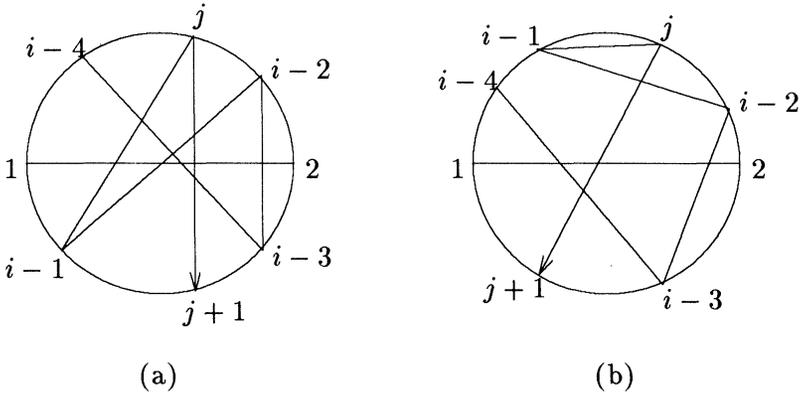


FIG. D. Case A22.

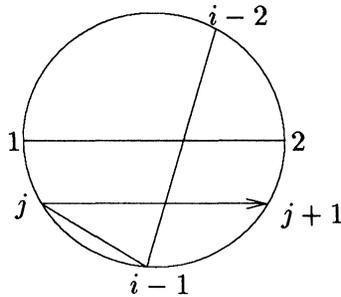


FIG. E. Case A23.

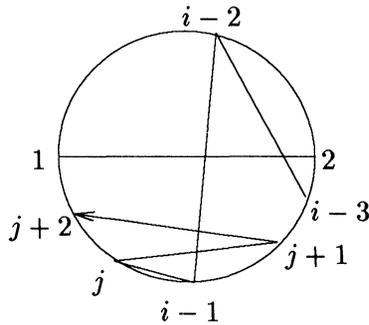


FIG. F. Case A232.

Justification. We translate $j+1$ counterclockwise until $1 < j+1 < i-2$, as in Fig. G(b). Now we may delete $i-1$ and j (in either order), and then delete $j+1$. Then $\langle 1, \dots, i-2, j+2 \rangle$ is a star path.

Case B. $(i-2, i-1, j)$ is a left turn. We have two subcases depending on whether index j lies in the arc from index $i-2$ clockwise to index 2, or from 2 clockwise to $i-1$.

Case B1. $j < 2 < i-2$. Push index j on the odd stack; thus we increment i and j by 1 each. See Fig. H.

Justification. $\langle 1, \dots, i-1, j \rangle$ is a star path.

Case B2. $j < i-1 < 2$. We consider subcases depending on $j+1$.

Case B21. $(i-1, j, j+1)$ is a right turn. We replace index $i-1$ by index j , and increment j by one.

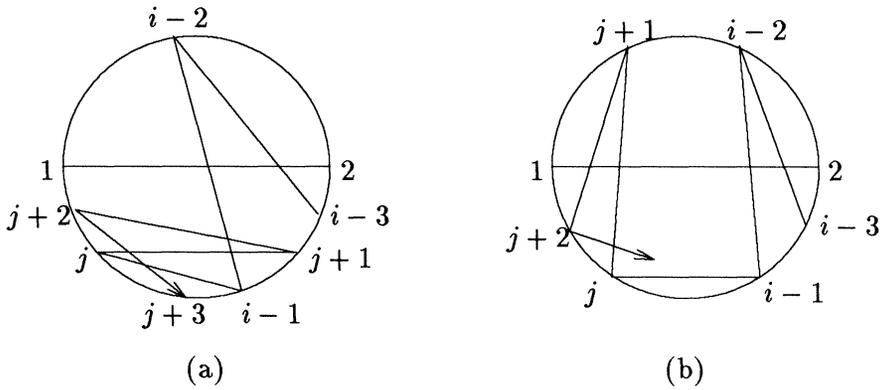


FIG. G. Case A2322.

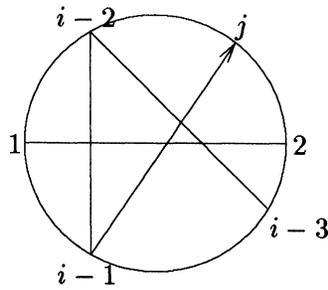


FIG. H. Case B1.

Justification. By a weak move counterclockwise from $i-3$ to some neighborhood of 2, we may assume that $2 < i-3 < j$. See Fig. J. After deleting $i-1, \langle 1, \dots, i-2, j \rangle$ is a star path.

Case B22. $(i-1, j, j+1)$ is a left turn. We consider two possible positions for index $j+1$.

Case B221. $j < j+1 < 1$. See Fig. K. We consider the position of index $j+2$ next.

Case B2211. $j < j+2 < 2$. After translating index j , we can push index j onto the odd stack and increment both i and j by one.

Justification. We can translate j counterclockwise until $i-2 < j < 2$. Now $\langle 1, \dots, i-1, j \rangle$ is a star path.

Case B2212. $2 < j+2 < j$. See Fig. L. We must examine index $j+3$ next.

Case B22121. $(j+1, j+2, j+3)$ is a left turn. Increment j by two.

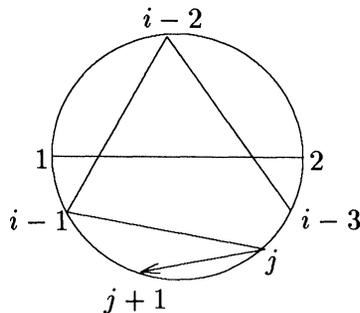


FIG. J. Case B21.

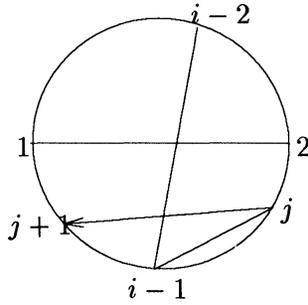


FIG. K. Case B221.

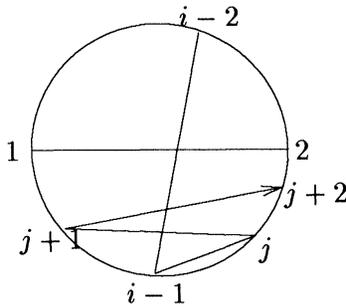


FIG. L. Case B2212.

Justification. We delete $j+1$ and then j .

Case B22122. $(j+1, j+2, j+3)$ is a right turn. After some translations, we may replace $i-1$ with j in the even stack, and push $j+1$ onto the odd stack. Effectively, we increment i by one and j by two.

Justification. See Fig. M(a). After making a weak move counterclockwise from index $i-3$ to some neighborhood of index 2, we may assume that $i-3 < j < i-1$. Then we can translate index $j+1$ clockwise until $i-2 < j+1 < 2$ and delete $i-1$. The sequence $\langle 1, \dots, i-2, j, j+1 \rangle$ is a star path (Fig. M(b)).

Case B222. $1 < j+1 < j$. After some translations, we can replace index $i-1$ by j on the stack; effectively we increment j by one.

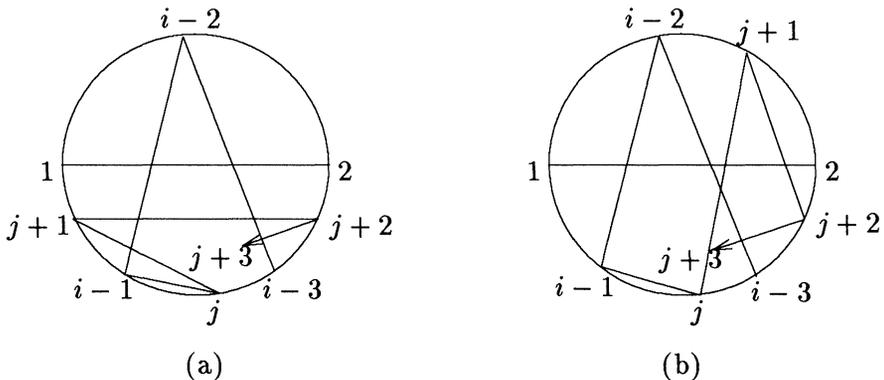


FIG. M. Case B22122.

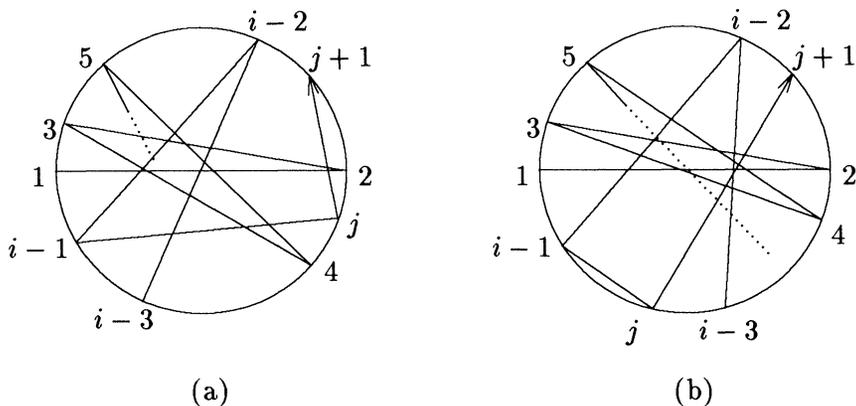


FIG. N. Case B222.

Justification. See Fig. N(a). We make a weak move counterclockwise from $i - 2$ to some small enough neighborhood of index 1 so that $i - 2 < j + 1 < j$. Next, make a weak move counterclockwise from $i - 3$ to some small enough neighborhood of index 2 so that $i - 3 < j < i - 1$. See Fig. N(b). Now delete index $i - 1$ and see that $\langle 1, \dots, i - 2, j \rangle$ is a star path.

This completes subcase B and hence the case where i is odd.

The case where i is even is similar and is left to the reader. Finally, when j reaches vertex $v_{n+2} = v_2$ again, we are done.

6. Conclusion. We have given a constructive analogue of the Whitney–Graustein theorem, resulting in a real-time algorithm to “untangle” any polygon. We emphasize that the true contribution of this work is the construction of the transformation steps: checking if two polygons are equivalent is in itself a trivial process of keeping a cumulative sum of the angles turned.

Our proof shows incidentally: (1) It suffices to use (T1), (T2) transformations to make a polygon irreducible, and (2) any two equivalent irreducible polygons are inter-transformable using only (T2) transformations.

Since the publication of these results, Vegter [4] has improved them by defining the *isothetic normal forms* for polygons, and showing that a linear number of regular transformation steps suffices to convert any polygon into its isothetic normal form.

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