# Recognizing String Graphs on Surfaces

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A surface (more formally, a 2-manifold) is a Hausdorff space in which every point has an open neighborhood homeomorphic to $\mathbb{R}^2$. A curve on a surface is a continuous map $\pi : [0,1] \to \Sigma$. An abstract graph $G = \{V, E\}$ is called a *string graph* on a surface $\Sigma$ if it is possible to assign a curve to each vertex of $G$ so that two curves intersect if and only if their correspondent vertices are adjacent in $G$. Grapham [2] introduced the problem of recognizing string graphs to the combinatorial community in 1976. Although similar problems had been suggested before (see [1] and [7]).

The other closely related problem is the *weak realization* of a graph. Let $G = \{V, E\}$ be an abstract graph, $R \subseteq \binom{E}{2} = \{\{e, f\} | e, f \in E\}$ a set of edge pairs and $\Sigma$ a two-manifold. We call a drawing $D$ of $G$ a weak realization of $(G, R)$ on $\Sigma$ if two edges $e, f \in E$ cross in $D$ only if $\{e, f\} \in R$. The string graph problem polynomially reduces to the weak realizability problem [3].

The decidability of both problems were unknown before 2001. Any upper bound on the number of intersections of edges in a weak realization or a string graph presentation would result in a brute force algorithm. Unexpectedly, Kratochvil and Matousek [3] found an exponential lower bound for the number of crossings. They also conjectured an upper bound of the form $2^{cn^k}$ for string graphs, where $n$ is the number of vertices and $c$ and $k$ are some constants.

Pach and Toth [4], and Shaefer and Stefankovic [6] independently showed both problems are in NEXP, by giving a proof for the upper bound conjecture. Later, Schaefer et al. [5] proved that the problem is actually in NP using LZ encoding of the possible solutions and reducing their lengths exponentially as a result. All steps of their algorithms also works for any surface, except the upper bound. So, they conjectured an upper bound of the form $2^{O(n^k)}$ for a constant $k$ and proved that the same problems on any surface is also in NP, assuming the correctness of their conjecture. Otherwise the problem is in NEXP for more complicated surfaces than the plane.

I propose working on their conjecture for the project of the course. Assuming their conjecture is correct we are looking for a proof of it. That proof will put the string graph problem for any surface in NP. It is also possible that we find a counterexample that shows the conjecture is not right.

## References


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