Project Proposal
Minimum Cuts in Directed Embedded Graphs

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1 Introduction

Both planar graphs and flows and cuts of transportation networks are classical areas in algorithms research. Recently, many results were extended from planar graphs to their natural generalization, higher genus surfaces. We are particularly interested in studying the minimum cut problem of graphs embedded on surfaces.

2 Grpahs and Surfaces

In this section we introduce the very basic concepts of graph embedding on surfaces. For more comprehensive treatments, see Mohar and Thomassen [1].

2.1 Surfaces and Curves

A *surface* Σ is a compact connected Hausdorff topological space that is locally homeomorphic to either the plane $\mathbb{R}^2$ or to the closed halfplane $\{(x,\ y) \in \mathbb{R}^2 \mid x \geq 0\}$. Formally, the surfaces are also known as *2-manifolds*. A surface is said to be with *boundary* if there exist points which have an open neighborhood homeomorphic to the closed halfplane. The set of such points forms the boundary of the surface. A surface is said to be *orientable* if one can consistently choose a normal vector for all points, and *non-orientable* otherwise. Equivalently, a surface is non-orientable if it contains a subset homeomorphic to the Mobius band.

A *path* p in a surface Σ is a continuous mapping from the closed interval $[0,\ 1]$ into Σ. A *loop* l is a path which maps the endpoints of the interval into the same point, called the *basepoint* of the loop. A *cycle* γ in Σ is a continuous mapping from the unit circle S^1 into Σ. An *arc* is path for which the image of the endpoints is on the boundary of Σ. Paths, loops, cycles, and arcs are refered as *curves*. The *genus* of a surface Σ is given by the maximum number of simple disjoint closed curves that one can draw on Σ without separating it.

2.2 Graph Embeddings

An *embedding* of an *undirected* graph $G = (V,\ E)$ on the surface Σ is: (i) a mapping of all vertices $v \in V$ into distinct points in Σ; and (ii) a mapping of all edges $(u,\ v) \in E$ into simple paths in Σ that only intersect at common endpoints. An embedding of a *directed* graph is an embedding mapping edges into oriented paths. The faces F of the embedding are the connected components of $\Sigma \setminus G$. An embedding is said to be *cellular* (or 2-cell) if all faces are homoemorphic to open disks. Let Σ be a surface of genus g with b boundaries. Then $|V| - |E| + |F| = \chi(\Sigma)$, where $\chi(\Sigma) = 2 - 2g - b$ if Σ is orientable and $\chi(\Sigma) = 2 - g - b$ otherwise.

3 Minimum Cuts

The *minimum cut* problem and its dual, the *maximum flow* problem are classical problems in analyzing transportation networks. Formally, a $(s,\ t) - cut\ C$ of a graph $G = (V,\ E)$ is a set of edges in E such that the vertex s and the vertex t are in different connected components in $(V,\ E \setminus C)$. A *minimum cut* $(s,\ t) - cut\ C$ is a $(s,\ t) - cut$ of minimum total weight. For planar graphs, there are algorithms computing the minimum cut that run in $O(n\log n)$ time both for undirected graphs and directed graphs [2], [3]. As a consequence of the Euler's formula, for any fixed surface Σ and any cellular embedded graph $G = (V,\ E)$, we have that $E = O(V)$. For arbitrary sparse graphs, the fastest known algorithm achieves a $O(n^2\log n)$ complexity [4], [5]. For integer capacities bounded by a constant U, the best result known is $O(n^{\frac{3}{2}}\log n\log U)$ [6]. Recent work by Chambers et al. allow the computation of the minimum cut of a graph embedded on an orientable surface of genus g in $g^{O(g)}n\log n$ time [7].

4 Proposed Work

To our knowledge, the problem of finding the minimum cut of an **undirected** weighted graph embedded on an orientable surface in near-liniar time is still open. We propose to develop an efficient solution.

References

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