CS 598: Computational Topology, Fall 2009
Project Proposal - Grid Minors in Map Graphs

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A key component in the proof of the Graph Minor Theorem is a Grid Minor Theorem, which states that any graph with treewidth at least some $f(r)$ contains an $r \times r$ grid as a minor [RS]. Grid Minor Theorems find applications in designing general classes of approximation and fixed parameter tractable algorithms. The best bound for $f(r)$ currently is 20^{2r^5}. Robertson, Seymour, Thomas [RST] showed every *planar* graph with treewidth r has an $\Omega(r) \times \Omega(r)$ grid as a minor. Demaine and Hajiaghayi [DH] showed that every graph excluding a fixed minor, H, with treewidth r has an $\Omega(r) \times \Omega(r)$ grid as a minor.

1 Map Graphs

Given an embedded planar graph, G, we partition the faces into two sets, $N(G), L(G)$, denoted *nations, lakes* respectively.

We define the *map graph* $M = M(G)$ as follows: M has a vertex for every nation of G and two vertices are adjacent in M if the corresponding nations in G share a vertex.

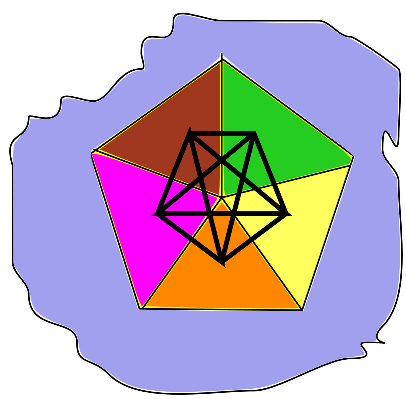

Figure 1. Example of a Map Graph. The underlying planar graph is a wheel with 5 spokes. The outside face (colored blue) is the single lake. The resulting map graph is K_5. The lower bound (below) is proved using the same graph, but with $r^2 - 1$ spokes.

Map graphs can have arbitrarily large cliques, and thus do not forbid any fixed minor H. Nevertheless, Demaine, Hajiaghayi and Kawarabayashi obtained an improved Grid Minor Theorem over Robertson-Seymour for general graphs.

Theorem 1. *[DHK] If the treewidth of the map graph M is r^3, then it has an $\Omega(r) \times \Omega(r)$ grid as a minor.*

This theorem implies the existence of fixed paramter tractable algorithms for computing certain graph parameters on map graphs. In particular, we call a parameter *minor-bidimensional* if it is at least $g(r)$ in the $r \times r$ grid graph and if the parameter does not increase when taking minors. One example of a minor-bidimensional parameter is the size of the smallest feedback vertex set. The following corollary gives algorithms for the general class of minor-bidimensional parameters.

Corollary 1. *Consider a parameter P that can be computed on a graph G in $h(w)n^{O(1)}$ time given a tree decomposition of G of width at most w. If P is minor-bidimensional and at least $g(r)$ in the $r \times r$ grid, then there is an algorithm computing P on any map graph G with running time $[h(O(g^{-1}(k))^7) + 2^{O([g^{-1}(k)]^7)}]n^{O(1)}$. In particular, if $h(w) = 2^{O(w)}$ and $g(k) = \Omega(k^2)$, then the running time is $2^{O(k^{7/2})}n^{O(1)}$*

There is also a lower bound for the specific family of map graphs.

Theorem 2. *[DHK] There are map graphs whose treewidth is $r^2 - 1$ and whose largest grid minor is $r \times r$.*

This is smaller than the Robertson, Seymour, and Thomas lower bound of $\Theta(r^2 \lg r)$, but that applies to general graphs and fails for map graphs.

Questions: Make the analysis for map graphs tight! Is there a map graph M where treewidth of M is $\Omega(r^3)$ but M contains no $r \times r$ grid as a minor? Such an example would improve the Robertson, Seymour, Thomas lower bound for general graphs.

Conjecture 1. *For some constant $c > 0$, every graph with treewidth at least cr^3 has an $r \times r$ grid minor. Furthermore, this bound is tight: some graphs have treewidth $\Omega(r^3)$ and no $r \times r$ grid minor.*

References:

[DH] Erik D. Demaine and MohammadTaghi Hajiaghayi, "Linearity of Grid Minors in Treewidth with Applications through Bidimensionality", Combinatorica, volume 28, number 1, January 2008, pages 1936.

[DHK] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Ken-ichi Kawarabayashi, "Algorithmic Graph Minor Theory: Improved Grid Minor Bounds and Wagner's Contraction", Algorithmica, to appear.

[RS] Neil Robertson and P. D. Seymour. Graph minors. V. Excluding a planar graph. Journal of Combinatorial Theory, Series B, 41:92 - 114, 1986.

[RST] N. Robertson, P. D. Seymour, and R. Thomas, Quickly excluding a planar graph, Journal of Combinatorial Theory, Series B, 62 (1994), pp. 323 - 348.