A key component in the proof of the Graph Minor Theorem is a Grid Minor Theorem, which states that any graph with treewidth at least some $f(r)$ contains an $r \times r$ grid as a minor [RS]. Grid Minor Theorems find applications in designing general classes of approximation and fixed parameter tractable algorithms. The best bound for $f(r)$ currently is $20^{2r^5}$. Robertson, Seymour, Thomas [RST] showed every planar graph with treewidth $r$ has an $\Omega(r) \times \Omega(r)$ grid as a minor. Demaine and Hajiaghayi [DH] showed that every graph excluding a fixed minor, $H$, with treewidth $r$ has an $\Omega(r) \times \Omega(r)$ grid as a minor.

1 Map Graphs

Given an embedded planar graph, $G$, we partition the faces into two sets, $N(G), L(G)$, denoted nations, lakes respectively.

We define the map graph $M = M(G)$ as follows: $M$ has a vertex for every nation of $G$ and two vertices are adjacent in $M$ if the corresponding nations in $G$ share a vertex.

![Figure 1. Example of a Map Graph. The underlying planar graph is a wheel with 5 spokes. The outside face (colored blue) is the single lake. The resulting map graph is $K_5$. The lower bound (below) is proved using the same graph, but with $r^2 - 1$ spokes.](image)

Map graphs can have arbitrarily large cliques, and thus do not forbid any fixed minor $H$. Nevertheless, Demaine, Hajiaghayi and Kawarabayashi obtained an improved Grid Minor Theorem over Robertson-Seymour for general graphs.

**Theorem 1.** [DHK] If the treewidth of the map graph $M$ is $r^3$, then it has an $\Omega(r) \times \Omega(r)$ grid as a minor.

This theorem implies the existence of fixed parameter tractable algorithms for computing certain graph parameters on map graphs. In particular, we call a parameter *minor-bidimensional* if it is at least $g(r)$ in the $r \times r$ grid graph and if the parameter does not increase when taking minors. One
example of a minor-bidimensional parameter is the size of the smallest feedback vertex set. The following corollary gives algorithms for the general class of minor-bidimensional parameters.

**Corollary 1.** Consider a parameter $P$ that can be computed on a graph $G$ in $h(w)n^{O(1)}$ time given a tree decomposition of $G$ of width at most $w$. If $P$ is minor-bidimensional and at least $g(r)$ in the $r \times r$ grid, then there is an algorithm computing $P$ on any map graph $G$ with running time $[h(O(g^{-1}(k))^7)+2^{O([g^{-1}(k)]^7)})]n^{O(1)}$. In particular, if $h(w) = 2^{O(w)}$ and $g(k) = \Omega(k^2)$, then the running time is $2^{O(k^7/2)}n^{O(1)}$

There is also a lower bound for the specific family of map graphs.

**Theorem 2.** [DHK] There are map graphs whose treewidth is $r^2 - 1$ and whose largest grid minor is $r \times r$.

This is smaller than the Robertson, Seymour, and Thomas lower bound of $\Theta(r^2 \lg r)$, but that applies to general graphs and fails for map graphs.

**Questions:** Make the analysis for map graphs tight! Is there a map graph $M$ where treewidth of $M$ is $\Omega(r^3)$ but $M$ contains no $r \times r$ grid as a minor? Such an example would improve the Robertson, Seymour, Thomas lower bound for general graphs.

**Conjecture 1.** For some constant $c > 0$, every graph with treewidth at least $cr^3$ has an $r \times r$ grid minor. Furthermore, this bound is tight: some graphs have treewidth $\Omega(r^3)$ and no $r \times r$ grid minor.

**References:**


